Uniform boundedness and spreading speed for some Reaction-diffusion SI systems

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Everything Disperses to Miami

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Introduction

Uniform bound

Propagation

Beyond the epidemic front
- The case $d = 1$
- The general case $d \neq 1$: averaging property

An epidemic model

We consider in this work the following epidemic model posed for $t > 0$ and $x \in \mathbb{R}^N$:

$$\begin{aligned} &\partial_t S - d\Delta S = \Lambda - \gamma S - \beta S I, \\ &\partial_t I - \Delta I = \beta S I - (\gamma + \mu) I, \\ &S(0,x) = S_0(x), \ \ I(0,x) = I_0, \end{aligned}$$

with $S(t,x)$ and $I(t,x)$ susceptible and infected individuals
$\Lambda > 0$ influx into the susceptible
$\gamma > 0$ natural death rate
$\mu > 0$ additional death rate due to the disease
$d > 0$ normalized diffusion coefficient ($d = d_S/d_I$)

ODE dynamics

The underlying ODE:

$$
\begin{aligned}
S' &= \Lambda - \gamma S - \beta S I, \\
I' &= \beta S I - (\gamma + \mu) I, \\
S(0) &= S_0, \quad I(0) = I_0 > 0,
\end{aligned}
$$

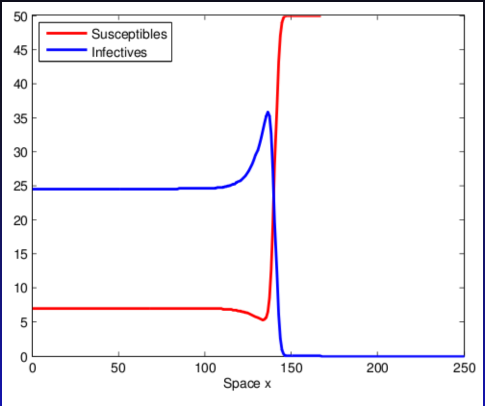
Epidemic threshold $R_0 = \frac{\beta\Lambda}{\gamma(\gamma+\mu)}$.

$R_0 \leq 1$	$R_0 > 1$
$\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} \to \begin{pmatrix} S_F \\ 0 \end{pmatrix} = \begin{pmatrix} \Lambda/\gamma \\ 0 \end{pmatrix}$	$\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} \to \begin{pmatrix} S_E \\ I_E \end{pmatrix} = \begin{pmatrix} S_F/R_0 \\ \frac{\gamma}{\beta}(R_0 - 1) \end{pmatrix}$

$R_0 > 1 \quad \Rightarrow$ Monostable dynamics

Spatial propagation of epidemics

We aim to describe the ability of this system to propagate the disease into a (initially) fully susceptible population.



One-dimensional travelling wave

When $N = 1$ and $R_0 > 1$ one can look at travelling wave solution for the model:

$$S(t,x) = U(x-ct), \quad I(t,x) = V(x-ct), \quad z = x - ct,$$

leading to the ODE system $z \in \mathbb{R}$:

$$\begin{cases} dU''(z) + cU'(z) = \Lambda - \gamma U(z) - \beta U(z)V(z), \\ V''(z) + cV'(z) = V(z)\left[\beta U(z) - (\gamma + \mu)\right] \\ (U,V)(\infty) = (S_F, 0), \quad (U,V)(-\infty) = (S_E, I_E). \end{cases}$$

Existence of waves

Existence AD and P. Magal with a system with additional age structure (and $d \geq 1$):

$$S_t - d\Delta S = \Lambda - \gamma S - S \int_0^\infty \beta(a) i da,$$

$$i_t + i_a = \Delta i - (\gamma + \mu) i \text{ and } i(t,0,x) = S(t,x) \int_0^\infty \beta(a) i(t,a,x) da$$

If $R_0 > 1$ and $\beta(a) \equiv \beta$ then existence of a family a solution for each speed $c > c^*$ minimal wave speed:

$$c^* = 2\sqrt{(\gamma + \mu)(R_0 - 1)}.$$

Expected dynamical propagation results

If $R_0 > 1$ (monostable dynamics) then for each S_0 bounded and $I_0 \not\equiv 0$ compactly supported then for each e$\in \mathbb{S}^{N-1}$:

$$\lim_{t\to\infty} \begin{pmatrix} S \\ I \end{pmatrix} (t, cte) = \begin{pmatrix} S_F \\ 0 \end{pmatrix}, \quad \forall c > c^*,$$

and

$$\lim_{t\to\infty} \begin{pmatrix} S \\ I \end{pmatrix} (t, cte) = \begin{pmatrix} S_E \\ I_E \end{pmatrix}, \quad \forall c \in [0, c^*),$$

Difficulties

• Uniform boundedness of the solution
Extension of Pierre's duality arguments for unbounded domain
• Lack of comparison principle
Dynamical system approach similar to uniform persistence to describe spatial propagation
• Description of the population after the epidemics

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Duality argument for bounded domain

Pierre's duality argument on bounded domain Ω is based on the following estimate: If on $(0,T)\times\Omega$

$$\partial_t w - d_1\Delta w + \theta_1 w \leq \theta_2\partial_t z + \theta_3\Delta z + \theta_4 z + H,$$

where $\theta_i\in\mathbb{R}$ and $H\in L^p_+(Q_T)$ and $p\in(1,\infty)$. Then there exists C_1 such that, for all $t\in(0,T]$:

$$C_1\|w^+\|_{L^p(Q_t)} \leq \|z\|_{L^p(Q_t)} + 1 + \int_0^t \|H(s)\|_{L^p(\Omega)}ds.$$

The whole space: Uniform Lebesgue spaces

L^p spaces are not well adapted to unbounded domain.
Make use of Uniform Lebesgue spaces $L^p_u\left(\mathbb{R}^N\right)$ defined by

$$L^p_u\left(\mathbb{R}^N\right) = \left\{\phi \in L^p_{loc}\left(\mathbb{R}^N\right) : \sup_{x\in\mathbb{R}^N} \|\phi\|_{L^p(B(x,1))} < \infty\right\}.$$

Heat semigroup on $L^p_u\left(\mathbb{R}^N\right)$ enjoys the following estimate

$$\|e^{\Delta t}\|_{\mathcal{L}\left(L^p_u, L^\infty_u\right)} \leq M\left(1 + t^{-\frac{N}{2p}}\right)$$

Estimates

Theorem

Let $T>0$, $h\in L^\infty$, $(\theta_1,\theta_2)\in\mathbb{R}^2$ and $\nu>0$ be given. Let $(u,v)\in W^{1,2}_\infty(0,T;\mathbb{R}^N)$ with $u\geq 0$ and

$$\left(\partial_t-\Delta+\nu\right)u(t,x)\leq h(t,x)+\left(\theta_1\partial_t+\theta_2\Delta\right)v(t,x).$$

Then for each $p\in(1,\infty)$, there exists some constant $C=C(p,\nu,N)>0$ s.t.

$$\left[\int_0^T\|u(t,.)\|^p_{L^p_u(\mathbb{R}^N)}dt\right]^{\frac{1}{p}}\leq K(p,T)\left[1+\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}+T^{\frac{1}{p}}\right],$$

with $K(p,T)>0$:

$$K(p,T)=C\left[1+\|h\|_{L^\infty((0,T)\times\mathbb{R}^N)}+\|v\|_{L^\infty((0,T)\times\mathbb{R}^N)}\left(|\theta_1|+|\theta_2|\right)\right].$$

Consequences for the SI-system

From the SI system on $\mathbb{R}^N$:

$$\begin{cases}(\partial_t - d\Delta)S = \Lambda - \gamma S - \beta SI \\ (\partial_t - \Delta)I = \beta SI - (\gamma + \mu)I,\end{cases}$$

Uniform bound for S is easy while

$$(\partial_t - \Delta)I + \mu I \leq \Lambda - \partial_t S + d\Delta S$$

$\hookrightarrow L^p_u$ control for I

$\Rightarrow$ Uniform bound due to parabolic regularity.

Consequences for the SI-system

Theorem

The system generates a strongly continuous semiflow $\{T(t)\}_{t\geq 0}$ *on* $\mathrm{BUC}_+(\mathbb{R}^2,\mathbb{R}^N)$ *and for each* $\kappa > 0$, *there exists* $\widehat{\kappa} > 0$ *such that*

$$\|T(t)U_0\|_\infty \leq \widehat{\kappa}, \ \ \forall t \geq 0, \ \forall U_0 \in \mathrm{BUC}_+(\mathbb{R}^2,\mathbb{R}^N),$$

with $\|U_0\|_\infty \leq \kappa$.

The case $R_0 \leq 1$

If $R_0 \leq 1$ then for each initial data, one has

$$\lim_{t \to \infty} (S, I)(t, x) = (S_F, 0),$$

uniformly for $x \in \mathbb{R}^N$.

The epidemic uniformly dies out.

Outer propagation

Assume $R_0 > 1$.
Similar to the Fisher-KPP equation:

Lemma

If $I_0 \not\equiv 0$ and compactly supported then for each $c \geq c^*$ and each $e \in \mathbb{S}^{N-1}$:

$$\lim_{t\to\infty} S(t, cte) = \frac{\Lambda}{\gamma},$$

and

$$\lim_{t\to\infty} I(t, cte) = 0.$$

Moving faster than the minimal wave speed: DFE.
Log phase can also be proved.

Inner propagation

Theorem

Assume $R_0 > 1$ and (S_0, I_0) positive and bounded with $I_0 \not\equiv 0$. Then there exists $\varepsilon > 0$ s.t. for each $c \in (-c^*, c^*)$, each $e \in \mathbb{S}^{N-1}$ and $x \in \mathbb{R}^N$:

$$\limsup_{t\to\infty} S(t, x + cte) \leq \frac{\Lambda}{\gamma} - \varepsilon,$$

and

$$\liminf_{t\to\infty} I(t, x + cte) \geq \varepsilon.$$

Slower than c^* the infection is persistent.

Weak spatial propagation

Assume that $R_0 > 1$. Let $\kappa > 0$ and $c_0 \in [0, c^*)$ be given. Then there exists $\varepsilon = \varepsilon(\kappa, c_0) > 0$ such that

$$\limsup_{t \to \infty} I(t, x + cte; U_0) \geq \varepsilon,$$

for each $x \in \mathbb{R}^N$, each $e \in \mathbb{S}^{N-1}$, each $c \in [-c_0, c_0]$ and $U_0 = (S_0, I_0)$ with $0 \leq S_0,\ I_0 \leq \kappa$ and $I_0 \not\equiv 0$.

Idea

If $I(t, x + cte) \approx 0$ then $S(t, x + cte) \approx \frac{\Lambda}{\gamma}$ and

$$(\partial_t - \Delta - ce.\nabla)\, I \approx \alpha I$$

with $\alpha > 0$ since $R_0 > 1$.
$\hookrightarrow$ exponentially growing sub-solution.

Then change the $\limsup$ into $\liminf$ by adapting the ideas of Freedman and Moson, Thieme, Magal and Zhao.

Inner propagation and uniformly persistent entire solutions

From inner propagation result:
Assume that $R_0 > 1$. Let $c \in (-c^*, c^*)$, $e \in \mathbb{S}^{N-1}$ and (S_0, I_0) with $I_0 \not\equiv 0$ be given. Let $\{t_n\}_{n \geq 0}$ be a given sequence tending to $+\infty$. Then (up to a subsequence)

$$\lim_{n \to \infty} (S, I)\,(t + t_n, x + c\,(t + t_n)\,e) = (S^\infty, I^\infty)\,(t, x - cet),$$

where (S^∞, I^∞) is a bounded entire solution s.t.

$$\inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} I^\infty(t, x) > 0$$

A uniformly persistence entire solution
It describes the population after the epidemic front.

Uniformly persistent entire solutions: case $d = 1$

Lemma

Assume $R_0 > 1$ and $d = 1$.
Let (S, I) be a uniformly persistence entire solution. Then one has:

$$(S, I)(t, x) \equiv (S_E, I_E).$$

If $R_0 > 1$ and $d = 1$, after the epidemic front, we only see the endemic stationary state.

Idea of the proof

Lyapunov like arguments:
Set $g(x) = x - 1 - \ln x$ and consider

$$W(t,x) = g\left(\frac{S(t,x)}{S_E}\right) + \frac{I_E}{S_E} g\left(\frac{I(t,x)}{I_E}\right).$$

Then W bounded and satisfies

$$(\partial_t - \Delta)\, W(t,x) \leq 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then show that $W \equiv 0$.

Time averaging

Theorem

Assume $R_0 > 1$ and let (S, I) be a uniformly persistent entire solution. Then one has for each continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{t-T}^{t+T} f\left[\begin{pmatrix} S \\ I \end{pmatrix}(s,x)\right] ds = f\begin{pmatrix} S_E \\ I_E \end{pmatrix},$$

uniformly with respect to $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Spatial averaging

Theorem

Assume $R_0 > 1$ and let (S, I) be a uniformly persistent entire solution. Then one has for each continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\lim_{R \to \infty} \frac{1}{(2R)^N} \int_{[-R,R]^N} f\left[\begin{pmatrix} S \\ I \end{pmatrix}(t, x+y)\right] dy = f\begin{pmatrix} S_E \\ I_E \end{pmatrix},$$

uniformly with respect to $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Idea: Time averaging

Consider the set

$$\mathcal{A} = \overline{\bigcup_{(t,h)\in\mathbb{R}\times\mathbb{R}^N} \begin{pmatrix} S(t,.+h) \\ I(t,.+h) \end{pmatrix}}^{C_{loc}}.$$

Compact metric separable space endowed with the weighted distance

$$d_{\mathcal{A}}(u,v) = \sup_{x\in\mathbb{R}^N} e^{-\|x\|} \left|u(x) - v(x)\right|.$$

Idea: Time averaging

Consider $\mathbb{M}(\mathcal{A})$ the set of probability measure on $\mathcal{A}$: Compact separable metric space endowed with the dual-bounded Lipschitz distance ($\sim w - C\left(\mathcal{A}\right)$ topology)

$$\pi\left(\mu,\nu\right) = \sup_{f \in \mathrm{Lip}\ (\mathcal{A}),\ \|f\|_{\mathrm{Lip}} \leq 1} \left| \int_{\mathcal{A}} f d\mu - \int_{\mathcal{A}} f d\nu \right|$$

Study the semiflow: $T^{\sharp} : [0,\infty) \times \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$:

$$T_t^{\sharp}\mu(B) = \mu\left(T_t^{-1}(B)\right).$$

Invariant measure

Lemma

The following holds true:

$$\mathbb{T}(\mathcal{A}) := \left\{\mu \in \mathbb{M}\left(\mathcal{A}\right) : \; T_t^\sharp \mu = \mu, \;\; \forall t \geq 0\right\} = \{\delta_{U_E}\}\,.$$

with $U_E = \begin{pmatrix} S_E \\ I_E \end{pmatrix}$

Lyapunov like arguments

For each $\mu \in \mathbb{T}(\mathcal{A})$ consider the map:

$$K\left[x,\mu\right] = \int_{\mathcal{A}} W\left(U(x)\right)\mu(dU),$$

with W defined by:

$$W\begin{pmatrix} S \\ I \end{pmatrix} = dg\left(\frac{I}{S_E}\right) + \frac{I_E}{S_E} g\left(\frac{I}{I_E}\right),$$

Then $K[.,\mu]$ is a bounded sub-harmonic map on $\mathbb{R}^N$, that is

$$\Delta K[x,\mu] \geq 0, \ \ \forall x \in \mathbb{R}^N, \ \mu \in \mathbb{T}(\mathcal{A}).$$

Lyapunov like argument

Looking at the points approaching $\sup K[., \mu]$ and maximum principle:
$\Rightarrow K[x, \mu] \equiv 0$.
Then $\mu\left(\{U_E\}\right) = 1$ and $\mu = \delta_{U_E}$.

Idea for the spatial averaging

Consider the set

$$\mathbb{P}(\mathcal{A}) = \left\{ \mu \in \mathbb{M}(\mathcal{A}) : \quad \sigma_h^\sharp \mu = \mu \; \forall h \in \mathbb{R}^N \right\}.$$

Then one has

Lemma

For each $\mu_0 \in \mathbb{P}(\mathcal{A})$:

$$\lim_{t \to \infty} \pi \left(T_t^\sharp \mu_0, \delta_{U_E} \right) = 0.$$

In other words, δ_{U_E} is globally stable for $T^\sharp$ restricted to $\mathbb{P}(\mathcal{A})$.