

# Asymptotic behavior of a nonlocal inhomogeneous equation

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EVERYTHING DISPERSES TO MIAMI  
Workshop in honor of Chris Cosner 60th birthday

## Evolution problem with non-local dispersion

Set  $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  nonnegative such that  $\int_{\mathbb{R}^N} K(x, y) dx = 1$  for all  $y \in \mathbb{R}^N$ , that is, for each  $y$ ,  $K(x, y)$  is a probability density in  $\mathbb{R}^N$ .

The equation

$$u_t(x, t) = \int_{\mathbb{R}^N} K(x, y) u(y, t) dy - u(x, t), \quad (1)$$

have been used to model diffusion processes in the following sense:

$u(y, t)$ : is thought of as a population density at location  $y$  at time  $t$ .

$K(x, y)$ : the probability that an individual at position  $y$  jumps from  $y$  to location  $x$ .

The rate at which individuals from all other places are arriving to location  $x$  is

$$\int_{\mathbb{R}^N} K(x, y) u(y, t) dy,$$

and the rate at which individuals are leaving location  $x$  to travel to all other places is

$$-\int_{\mathbb{R}^N} K(y, x) u(x, t) dy = -u(x, t).$$

In the absence of external sources or reaction terms this implies that the density function  $u$  must satisfy the above equation.

A more specific dispersal model that has been treated by several authors in different contexts, is the case when  $K$  is a convolution kernel.

More precisely,

$$K(x, y) = G(x - y)$$

where  $G : \mathbb{R}^N \rightarrow \mathbb{R}$  is a nonnegative function such that  $\int_{\mathbb{R}^N} G(y) dy = 1$ .

An important case, is when  $G(x) = J(|x|)$ , with  $J : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\int_{\mathbb{R}^N} J(|x|) dx = 1$ . In this situation we have that the probability that an individual in  $y$  jumps to  $x$  only depends on the distance between  $y$  to  $x$ .

In this situation, the equation modeling the dispersal of a population that moves according to such assumption is given by

$$u_t(x, t) = \int_{\mathbb{R}^N} J(|x - y|) u(y, t) dy - u(x, t). \quad (2)$$

This equation shares many properties with the heat equation:

$$u_t = d\Delta u, \quad (3)$$

where  $\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$  and  $d > 0$  is the diffusion coefficient.

- The only positive steady states are the constant functions.
- Infinite speed of propagation.
- Maximum principle.

But, equation (2) does not have a regularizing effect,  $u(\cdot)$  has the same regularity that its initial condition  $u_0$ .

In the work of Chasseigne, Chaves and Rossi (2006) it is shown that under some appropriate conditions for the Fourier transform of  $J$ , the solutions of (3) and (2) with initial condition with finite mass, both have the same decay as those of (3) as  $t \rightarrow \infty$ , moreover the solutions of (2) behave as Gaussians when  $t \rightarrow \infty$ .

We will consider a variant of (2). Our main hypothesis is that the probability of jumping from one point to others depends on the points by means of a [step size](#), which is given by a positive function  $g(y)$ .

More precisely,

$$K(x, y) = J \left( \frac{|x - y|}{g(y)} \right) \frac{1}{g(y)^N}.$$

Observe that

$$\int_{\mathbb{R}^N} J \left( \frac{|x - y|}{g(y)} \right) \frac{1}{g(y)^N} dx = 1,$$

as required.

Then equation (1) becomes

$$u_t(x, t) = \int_{\mathbb{R}^N} J \left( \frac{|x - y|}{g(y)} \right) \frac{u(y, t)}{g(y)^N} dy - u(x, t).$$

## Basic Assumptions

Within this talk we will assume that  $N = 1$ . Concerning  $J$  and  $g$  the basic assumptions are

(J1)  $J(|x|) > 0$  for all  $|x| \leq 1$ , and  $J(|x|) = 0$  if  $|x| \geq 1$ , that is the support of  $J$  is  $[-1, 1]$

(J2)  $J$  Hölder continuous

(g1)  $g$  is positive, bounded below and continuous

(g2) For all  $x$  the set  $\{y : |x - y| \leq g(y)\}$  is bounded

Observe that by (J1) the individuals at location  $y$  are not allowed to jump off the unit ball centered at  $y$  and radius  $g(y)$  since

$$J\left(\frac{|x - y|}{g(y)}\right) \neq 0 \text{ if and only if } |x - y| < g(y)$$

## Problem

Our purpose is to study the asymptotic behavior of the solutions of the initial value problem

$$u_t(x, t) = \int_{\mathbb{R}} J\left(\frac{|x-y|}{g(y)}\right) \frac{u(y, t)}{g(y)} dy - u(x, t) \quad \text{in } \mathbb{R} \times [0, \infty), \quad (4)$$

with a prescribed initial data

$$u(x, 0) = u_0(x) \quad \text{on } \mathbb{R}$$

which we suppose with finite mass, i.e

$$\int_{\mathbb{R}} |u_0(x)| dx < \infty.$$

We will see that to understand the asymptotic behavior of (4) it is crucial to understand the positive solutions for the stationary problem:

$$p(x) = \int_{\mathbb{R}} J\left(\frac{|x-y|}{g(y)}\right) \frac{p(y)}{g(y)} dy \quad \text{in } \mathbb{R}. \quad (5)$$

We recall that if  $g = 1$  the only positive solutions of (5) are the constants.

## Basic facts

### Existence

Given  $u_0 \in L^1(\mathbb{R})$  there exists a unique solution  $u \in C^1(\mathbb{R}, L^1(\mathbb{R}))$  of (4). The solution  $u$  conserves the total mass, that is

$$\int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} u_0(x) dx$$

for all  $t > 0$ .

### Infinite speed of propagation

If  $u_0 \in L^1(\mathbb{R})$  is nonnegative a.e., then  $u(x, t) \geq 0$  a.e. in  $\mathbb{R}$  for each  $t \geq 0$ . If in addition  $u_0 \not\equiv 0$ , then  $u(x, t) > 0$  a.e. in  $\mathbb{R}$  for all  $t > 0$ .



# Energy

Suppose that (5) admits a positive solution  $p$  (continuous). With the aid of this equilibrium, we can construct an energy functional following ideas from [Michel, Mischler and Perthame (2005)].

We define

$$E(t) = \int_{\mathbb{R}} \frac{u^2}{p} dx.$$

We suppose that the energy is well defined, then differentiating we obtain:

$$E'(t) = - \int_{\mathbb{R}} \int_{\mathbb{R}} J \left( \frac{|x-y|}{g(y)} \right) \frac{p(y)}{g(y)} \left[ \frac{u}{p}(t, x) - \frac{u}{p}(t, y) \right]^2 dy dx.$$

So the energy is decreasing along the orbits. Observe that if we can prove convergence  $u(x, t) \rightarrow v(x)$  as  $t \rightarrow \infty$  then we must have

$$\frac{v}{p}(x) - \frac{v}{p}(y) = 0, \quad \text{for } x, y$$

that is  $v/p = c$ .

We remark that the same computation is true when  $p$  is a positive continuous supersolution, i.e.

$$p(x) \geq \int_{\mathbb{R}} J \left( \frac{|x-y|}{g(y)} \right) \frac{p(y)}{g(y)} dy \quad \text{in } \mathbb{R}$$

## Theorem

Suppose that (5) has a positive solution  $p$  (or a continuous supersolution  $p \notin L^1(\mathbb{R})$ ).

- 1 If  $p \notin L^1(\mathbb{R})$ , then if  $u$  is a solution of (4), we have that  $u(\cdot, t) \rightarrow 0$  in  $L^1_{loc}(\mathbb{R})$  as  $t \rightarrow \infty$ .
- 2 If  $p \in L^1(\mathbb{R})$ , then  $u(\cdot, t) \rightarrow \lambda p$  in  $L^1(\mathbb{R})$  as  $t \rightarrow \infty$ , with  $\lambda = \frac{\int u_0}{\int p}$ .

The idea is to prove the results first assuming that the initial value  $u_0 \in X_p \cap L^1(\mathbb{R})$ , where

$$X_p = \left\{ u \in C(\mathbb{R}) : \sup_{x \in \mathbb{R}} \frac{|u(x)|}{p(x)} < \infty \right\},$$

and then proceed by density since  $X_p \cap L^1(\mathbb{R})$  is dense in  $L^1(\mathbb{R})$  and using the following inequality which holds for any solution of (4):

$$\|u(\cdot, t) - v(\cdot, t)\|_1 \leq \|u_0 - v_0\|_1$$

To prove the result for initial conditions in  $X_p \cap L^1(\mathbb{R})$  we proceed as follows:

- 1 If we define  $Tu = \int_{\mathbb{R}} J \left( \frac{|x-y|}{g(y)} \right) \frac{u(y)}{g(y)} dy$  then  $T : X_p \rightarrow X_p$ , and if  $u_0 \in X_p$  we have  $|u(\cdot, t)|_{X_p} \leq |u_0|_{X_p}$  for all  $t$ . In particular  $|u(x)| \leq |u_0|_{X_p} p(x)$ .
- 2 If  $u_0 \in L^1(\mathbb{R}) \cap X_p$  then  $E(t) \leq |u_0|_{X_p} \|u_0\|_1$ .
- 3  $E(t)$  is  $C^2(\mathbb{R})$  and  $E', E''$  are bounded, which since  $E' \leq 0$  implies that  $E' \rightarrow 0$ .
- 4 Using the hypothesis on  $g$  and  $J$  we can show that if  $t_n \rightarrow \infty$  then up to a subsequence  $u(\cdot, t_n) \rightarrow v$  locally uniformly in  $C(\mathbb{R})$ .
- 5 We can pass to the limit and conclude.

So to characterize the asymptotic behavior of the solutions we should study the existence and behavior of the positive solutions of (5)

To construct positive solutions of (4) we can proceed by solving the problem in a finite interval and then passing to the limit. Using Krein Rutman we can find a positive continuous solution  $p_K$  of

$$\int_{-K}^K J\left(\frac{|x-y|}{g(y)}\right) \frac{p_K(y)}{g(y)} dy = \int_{-K}^K J\left(\frac{|x-y|}{g(x)}\right) \frac{p_K(x)}{g(x)} dy,$$

with  $p_K(0) = 1$ .

Set  $L > 0$  fixed, then  $g$  is bounded in  $[-L, L]$ . Then for  $K$  large enough we have that  $p_K$  is a solution of the equation in  $[-L, L]$ . We can also prove bounds for  $p_K$  in  $[-L, L]$  and pass to the limit to obtain a nontrivial positive solution.

## Bounded steady states

In the case when  $g$  is bounded above we have the following result:

### Theorem

(Cortázar, Coville, Elgueta, M, JDE 2007) Suppose that  $a < g < b$ , there exists  $0 < \alpha_1 < \alpha_2 < \infty$  such that the equilibrium  $p$  constructed satisfies  $\alpha_1 < p(x) < \alpha_2$  in  $\mathbb{R}$ . Moreover  $p$  is the only positive solution with  $p(0) = 1$ .

To prove this result we need to use the following lemma:

### Lemma

For any solution of  $p$  of (5), the function  $f(D)$  given by

$$f(x) = \int_0^b \int_{x-w}^{x+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z) dz ds dw \quad (6)$$

satisfies  $f''(x) = 0$ . In particular, if  $p$  is positive we have that  $f(x)$  is constant.

- ① We remark that under suitable assumptions we can also construct solutions that change sign, corresponding to  $f(x) = x$  (Cortázar, Elgueta, García-Melián, M, Indiana Math. J. 2011). These solutions behave as linear functions for  $|x|$  large. But, even the equation  $J * u - u = 0$  may have solutions which are oscillatory with exponential growth.
- ② Formula (6) is true for  $g$  unbounded if the last integral is in a finite interval

$$f(x) = \int_0^\infty \int_{x-w}^{x+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z) dz ds dw.$$

For example when  $g(s) < s$  for  $|s|$  large.

## Theorem

Assume  $g$  is continuous bounded above and below. There exists a solution  $q$  of (5) satisfying

$$f_q(x) = x, \quad (7)$$

for every  $x \in \mathbb{R}$  and

$$a_1 \leq \frac{q(x)}{x} \leq a_2, \text{ as } |x| \rightarrow \infty, \quad (8)$$

for some  $a_1, a_2 > 0$ .

Concerning uniqueness we have the following result:

## Theorem

Assume that  $J, g$  are as before and let  $u$  be a solution of (5) verifying  $|u(x)| \leq C(|x|^k + 1)$  in  $\mathbb{R}$  for some  $C > 0$  and  $k > 0$ . If  $F_u(x) = Ax + B$  then  $u = Aq + Bp$ , with  $q$  as above and  $p$  positive solution of (5).

The requirement that  $u$  has polynomial growth is not a technical assumption. In the particular case where  $g = 1$ , we will construct a kernel  $J$  such that (5) has solutions with exponential growth, which verify

$$F_u(x) = 0.$$

### Theorem

*Let  $g = 1$ . Then there exists a  $C^1$ , nonnegative, compactly supported kernel  $J$  with unit integral such that (5) admits solutions of the form*

$$u(x) = e^{ax} \cos(bx), \quad u(x) = e^{ax} \sin(bx)$$

*for  $a, b \neq 0$ . Moreover,  $f_u(x) = 0$  when  $x \in \mathbb{R}$ .*



## Solutions in $L^1$

We have the following result

### Theorem

Suppose that  $g(y) \leq |y|$  for  $y$  large and

$$\liminf_{|y| \rightarrow \infty} \frac{g(y)}{|y|^\alpha} > 0,$$

for some  $\alpha > \frac{1}{2}$ . Then all the solutions of (5) are in  $L^1(\mathbb{R})$ .

In contrast when  $g(y) \sim |y|^\alpha$  for  $|y|$  large and  $\alpha < 1$  we have solutions, but they do not belong to  $L^1$ .

### Theorem

Suppose that

$$0 \leq \liminf_{|y| \rightarrow \infty} \frac{g(y)}{|y|^\alpha} \leq \limsup_{|y| \rightarrow \infty} \frac{g(y)}{|y|^\alpha} < \infty$$

with  $0 \leq \alpha \leq \frac{1}{2}$ , then (5) has a bounded positive solution which is not in  $L^1$ .

## Proof of existence of solutions in $L^1$

We give a sketch of the proof of the first theorem in the case when  $g(y) = |y|^\alpha$  with  $\alpha > \frac{1}{2}$ . We suppose also that we have a solution. In this situation we will estimate

$\int_N^\infty \rho$  and  $\int_{-\infty}^{-N} \rho$  using formula (6).

Assume that  $x > 0$  large. Observe that we always integrate when

$$\frac{w}{g(s)} < 1, \quad s \in (x - w, x + w),$$

and by our hypothesis, we can prove that  $w \leq Cx^\alpha$ . So we can assume that

$$1 = \int_0^{Cx^\alpha} \int_{x-w}^{x+w} \rho(s) \int_{\frac{w}{g(s)}}^1 J(z) dz ds dw.$$

Set  $\delta > 0$  small, we have that

$$\begin{aligned} 1 &\geq \int_{\frac{\delta x^\alpha}{2}}^{\delta x^\alpha} \int_x^{x+w} \rho(s) \int_{\frac{w}{g(s)}}^1 J(z) dz ds dw \\ &\geq \int_{\frac{\delta x^\alpha}{2}}^{\delta x^\alpha} \int_x^{x+w} \rho(s) \int_{2\delta}^1 J(z) dz ds dw \\ &\geq Cx^\alpha \int_x^{x+\frac{\delta x^\alpha}{2}} \rho(s) ds. \end{aligned}$$

Thus,

$$\int_x^{x+\frac{\delta x^\alpha}{2}} p(s) ds \leq \frac{C}{x^\alpha}.$$

Set a sequence given by  $x_{n+1} = x_n + \frac{\delta x_n^\alpha}{2}$  we want to prove  $x_n \sim n^\beta$  with  $\beta > 0$ . For this to happen we need

$$n^\beta + \frac{\delta n^{\beta\alpha}}{2} \sim (n+1)^\beta, \quad (n+1)^\beta - n^\beta \sim \frac{\delta n^{\beta\alpha}}{2}.$$

So we need  $\beta = \frac{1}{1-\alpha}$  so

$$\int_1^\infty p(s) ds \leq \sum_{n=1}^\infty \frac{1}{n^{\beta\alpha}}$$

So we need  $\beta\alpha > 1$  for the sum to be finite, i.e.  $\frac{\alpha}{1-\alpha} > 1$  which happens when  $\alpha > 1/2$ .

## Nonexistence of positive solutions in $L^q$

The equation (5) does not always have positive solutions. We will provide some examples when this is the case. First, we have that when

$$\int_{\mathbb{R}} \frac{1}{g} < \infty$$

then the operator  $T : L^q(\mathbb{R}) \rightarrow L^q(\mathbb{R})$  is compact for all  $1 < q < \infty$ . We note that this is the case if  $g(x) = |x|^\kappa$  with  $\kappa > 1$  for  $|x|$  large. But, then for all  $x$  the set  $\{y : |x - y| \leq g(y)\}$  is unbounded. Let's prove that under the above assumption and  $\lim_{|x| \rightarrow \infty} g(x) = \infty$  that the equation  $Tu = u$  does not have any solution in  $L^q$ .

### Theorem

*If  $\int_{\mathbb{R}} \frac{1}{g} < \infty$  and  $g(y) \rightarrow \infty$  as  $|y| \rightarrow \infty$ , then there are no solutions in  $L^1$  to (5). Moreover in this case, (5) has a positive supersolution which is not in  $L^1$ .*

If not, then the spectral radius  $r \geq 1$ . Then there exists  $v(x) \in L^q$  such that  $T^*v = rv$ , it can be proved that  $v$  positive and continuous.

Then since

$$rv(x) = \int_{\mathbb{R}} J\left(\frac{|x-y|}{g(x)}\right) \frac{v(y)}{g(x)} dy,$$

we can use Jensen's inequality to prove that  $v(x) \leq \frac{C}{g(x)}$ , thus  $v(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then  $g$  reaches its maximum in  $\mathbb{R}$  and that is a contradiction to the fact  $r \geq 1$ .

Thus,  $r < 1$  and there are no solutions in  $L^q$ . Also there are no solutions  $p$  in  $L^1$ , otherwise we would have  $p(x) \leq \frac{C}{\min g}$  which implies  $p \in L^\infty \cap L^1 \subset L^q$ .

To construct the supersolution we can do it by solving  $u - Tu = h$  with  $h > 0$ ,  $h \in L^q$ . We have  $u > 0$  since  $u = \sum_{n=0}^{\infty} T^n u$ .

## Nonexistence of steady states

### Theorem

Suppose that there exists constants  $1 < \alpha_1 < \alpha_2$ ,  $C_1, C_2 > 0$  such that  $C_1|x|_1^{\alpha_1} \leq g(x) \leq C_2|x|_2^{\alpha_2}$  for  $|x|$  large. Then (5) does not have any positive solution.

Let's show the result in the simple case  $g(y) = |y|^\alpha$  with  $\alpha > 1$  for  $|y|$  large. Suppose we have a solution. Then, in this situation we must have that  $p/g \in L^1(\mathbb{R})$ . Take  $|x|$  large, in this situation  $|x - y| \leq g(y)$  implies that  $|y| > c|x|^{1/\alpha}$  with  $c$  constant, then

$$p(x) = \int_{\mathbb{R}} J \left( \frac{|x - y|}{g(y)} \right) \frac{p(y)}{g(y)} dy \leq \|J\|_\infty \int_{|y| \geq c|x|^{1/\alpha}} \frac{p(y)}{g(y)} dy,$$

from where  $p(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  and then  $p$  is bounded. Moreover, using the same inequality we obtain that for  $|x|$  large

$$p(x) \leq C \int_{|y| \geq c|x|^{1/\alpha}} \frac{1}{g(y)} dy \leq \frac{C}{|x|^{\frac{\alpha-1}{\alpha}}},$$

thus  $p \in L^q(\mathbb{R})$  with  $q > \frac{\alpha}{\alpha-1}$  which is a contradiction.