PHD QUALIFYING EXAM IN TOPOLOGY

1. Let X, Y and Z be topological spaces and $f : X \to Y$ and $g : Y \to Z$ continuous maps.

(a) Define the group $C_n(X)$ of singular n–chains in X

(b) Define the homomorphism $f_\# : C_n(X) \to C_n(Y)$

(c) Show that $(g \circ f)_\# = g_\# \circ f_\#$

(d) Define the group $C^n(X)$ of singular n–cochains in X

(e) Define the homomorphism $f^\# : C^n(Y) \to C^n(X)$

2. Given a short exact sequence of chain complexes and chain maps,

$$0 \to A \to B \to C \to 0,$$

describe the boundary homomorphism $\partial : H_n(C) \to H_{n-1}(A)$.

3. The cell complex X is obtained from two copies of $\mathbb{R}P^2$ by identifying $\mathbb{R}P^1 \subset \mathbb{R}P^2$ in one copy of $\mathbb{R}P^2$ with $\mathbb{R}P^1 \subset \mathbb{R}P^2$ in the other copy of $\mathbb{R}P^2$.

(a) Calculate the fundamental group $\pi_1(X, x)$

(b) Calculate the homology groups $H_*(X; \mathbb{Z})$

(c) Calculate the cohomology groups $H^*(X; \mathbb{Z}/2)$

4. Let $p : \tilde{X} \to X$ be a covering map between connected cell complexes.

(a) State the homotopy lifting property for p

(b) Prove or disprove: $p_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective

(c) Prove or disprove: $p_* : H_*(\tilde{X}; \mathbb{Z}) \to H_*(X; \mathbb{Z})$ is injective.

5. Let X be a closed connected 3–manifold which is non–orientable. Show that $H_1(X; \mathbb{Z})$ is infinite.

6. (a) Define the degree of a map $f : X \to Y$ of closed connected oriented n–dimensional manifolds, $n \geq 1$.

(b) Show that the degree of any map $f : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$ is even.

7. Describe the Hurewicz homomorphism $h : \pi_n(X) \to H_n(X; \mathbb{Z})$ and state the Hurewicz theorem.