Qualifying Exam - Algebraic Topology
May 27th, 2021

Do all six problems
You may use standard formulae for computations.

(1) (a) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all degrees.
(b) Show that they are not homotopy equivalent in two ways:
(i) by considering the homology groups of their universal covers
(ii) by considering their cohomology rings.

(2) (a) Compute the fundamental groups of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$.
(b) Show that $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is not a retract of $\mathbb{R}P^2 \times \mathbb{R}P^2$.
(c) Show that any map $f\colon \mathbb{R}P^2 \vee \mathbb{R}P^2 \to S^1$ is null-homotopic.

(3) Let F be the complement of an open disk in the torus $S^1 \times S^1$. Determine all connected surfaces that arise as 3–fold covers of F.

(4) Let Y be a CW complex where

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, \text{ or } 4 \\ \mathbb{Z}/12\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the cohomology groups $H^*(Y;\mathbb{Z})$ and $H^*(Y;\mathbb{Z}/2\mathbb{Z})$.

(5) Suppose M is a compact orientable n–manifold whose boundary ∂M is a rational homology sphere, i.e. $H_*(\partial M;\mathbb{Q}) = H_*(S^{n-1},\mathbb{Q})$. Use Poincare Duality to show that M has Euler characteristic $\chi(M) = 1$.

(6) Let $Y = \mathbb{R}P^3/\mathbb{R}P^1$. That is, Y is the quotient space $\mathbb{R}P^3/\sim$ where the equivalence relation $\sim$ collapses the standard $\mathbb{R}P^1 \subset \mathbb{R}P^3$ to a single point.
(a) Determine the integral homology of the pair $(\mathbb{R}P^3, \mathbb{R}P^1)$ and of the space Y.
(b) Compute $\pi_1(Y)$.

$$Hom(A,B) = \begin{array}{c|cc} A\backslash B & \mathbb{Z} & \mathbb{Z}/m \\ \hline \mathbb{Z} & \mathbb{Z} & \mathbb{Z}/m \\ \mathbb{Z}/n & 0 & \mathbb{Z}/\gcd(m,n) \end{array} \qquad Ext(A,B) = \begin{array}{c|cc} A\backslash B & \mathbb{Z} & \mathbb{Z}/m \\ \hline \mathbb{Z} & 0 & 0 \\ \mathbb{Z}/n & \mathbb{Z}/n & \mathbb{Z}/\gcd(m,n) \end{array}$$

$$Tor(A,B) = \begin{array}{c|cc} A\backslash B & \mathbb{Z} & \mathbb{Z}/m \\ \hline \mathbb{Z} & 0 & 0 \\ \mathbb{Z}/n & 0 & \mathbb{Z}/\gcd(m,n) \end{array}$$