PRELIMINARY EXAM IN TOPOLOGY

AUGUST 2007

1. Let $f : X \to Y$ be a map of topological spaces. Prove that f is continuous if and only if, for every $x \in X$ and every open neighborhood $V(f(x)) \subset Y$, there exists an open neighborhood $U(x) \subset X$ such that $f(U(x)) \subset V(f(x))$.

2. Let (X, d) be a metric space. Prove that:

(a) For any subset $A \subset X$, the formula $f_A(x) = \inf\{\, d(x,a) \mid a \in A \,\}$ defines a continuous function $f_A : X \to \mathbb{R}$.

(b) $\bar{A} = \{\, x \in X \mid f_A(x) = 0 \,\}$ (here $\bar{A}$ stands for the closure of A in X).

3. Prove that a compact subset of a Hausdorff space is always closed.

4. Let X be a connected normal space, and let A be a proper closed subset of X. Show that there is a continuous surjective map $f : X \to S^1$ such that $f(a) = 1$ for all $a \in A$ (here, S^1 is the unit circle $|z| = 1$ in the complex plane).

5. For $x, y \in \mathbb{R}^2 - \{0\}$, put $x \sim y$ if and only if x and y lie on a line through the origin. Show that $\mathbb{R}^2 - \{0\}/\sim$ with the quotient topology is homeomorphic to a circle.

6. Show that if $f : X \to Y$ is an immersion of smooth manifolds then its differential $df : TX \to TY$ is also an immersion.

7. Prove that no smooth compact manifold of dimension $n \geq 1$ is contractible.

8. Let C be a simple closed curve in the xy–plane not passing through the origin. Evaluate

$$\oint_C \frac{x\,dy - y\,dx}{x^2 + y^2}$$

(make sure your answer covers all possible cases).