PRELIMINARY EXAM

AUGUST 2008

1. Let (X, d) be a metric space, $A \subset X$, and $x \in X$. Prove that a point x is in the closure of A if and only if there is a sequence $x_n \in A$ which converges to x.

2. Prove that every continuous map $f : [0, 1] \to [0, 1]$ has a fixed point. Is the same true for continuous maps $(0, 1) \to (0, 1)$?

3. Let $\mathbb{R}_\ell$ be the real line with the lower limit topology, that is, topology whose basis consists of all half-open intervals of the form $[a, b)$.

(1) Determine the path components of $\mathbb{R}_\ell$.
(2) Describe all continuous functions $f : \mathbb{R} \to \mathbb{R}_\ell$, where $\mathbb{R}$ stands for the real line with the standard topology.

4. Let $f : X \to Y$ be a continuous bijection of a compact space X to a Hausdorff space Y. Show that f is a homeomorphism.

5. For which values of a does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2+y^2+z^2 = a$ transversely ? What does the intersection look like for different values of a ?

6. The orthogonal group $O(n)$ consists of all $n \times n$ matrices A such that $AA^t = E$, where E is the identity matrix.

(1) Evaluate the tangent space to $O(n)$ at $E \in O(n)$.
(2) Prove that $O(n)$ is compact.

7. Let $f : S^2 \to S^2$ be a smooth map.

(1) Can a point $q \in S^2$ have infinitely many preimages ?
(2) Prove that the set of points $q \in S^2$ with infinitely many preimages has measure zero.
(3) Prove that, if f is not surjective, it is homotopic to a constant map.

8. Show that the vector field on $\mathbb{R}^2 - \{(0, 0)\}$ given by

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

has zero curl but cannot be written as the gradient of any smooth function $f : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}$.