MS TOPOLOGY QUALIFYING EXAM
AUGUST 2009

(1) Given a set X, let $\mathcal{T}$ be the collection of all subsets U of X such that $X - U$ is either finite or all of X. Show that $\mathcal{T}$ is a topology on X.

(2) Prove or give a counterexample: A continuous bijective map from one topological space to another is a homeomorphism.

(3) A subspace $A \subset \mathbb{R}^n$ has the property that every continuous function $f\colon A \to \mathbb{R}$ attains its maximum on A. Show that A must be compact.

(4) For points $x, y \in \mathbb{R}^2 - \{0\}$ impose $x \sim y$ if and only if x and y lie on the same line through the origin of $\mathbb{R}^2$. Show that $\mathbb{R}^2 - \{0\}/\sim$ with the quotient topology is homeomorphic to a circle.

(5) Prove or give a counter example: A smooth homeomorphism from one smooth manifold to another is a diffeomorphism.

(6) Prove that the set X of all non-zero 2×2 real matrices A such that AA^t is diagonal is a submanifold of $\mathbb{R}^4 = M_{2\times 2}(\mathbb{R})$. What is the dimension of X? Is X compact? (A^t denotes the transpose of A.)

(7) Prove there is no non-vanishing vector field on the sphere S^2. On what connected, closed, compact, orientable surfaces does a non-vanishing vector field exist? What about for surfaces with boundary?

(8) Suppose Z_0 and Z_1 are compact, cobordant, p-dimensional smooth submanifolds of a smooth manifold X. Prove that

$$\int_{Z_0} \omega = \int_{Z_1} \omega$$

for every closed p-form ω on X.