# MS TOPOLOGY PRELIMINARY EXAM

AUGUST 2011

(1) Let $A$ be a subset of a topological space $X$. Prove that $A$ is open in $X$ if and only if every point in $A$ has a neighborhood contained in $A$.

(2) Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$ with itself.
   A. Define both the *product topology* and the *box topology* on $\mathbb{R}^\omega$.
   B. Consider the function $f\colon \mathbb{R} \to \mathbb{R}^\omega$ given by $f(t) = (t, t, t, \dots)$. Prove that $f$ is continuous when $\mathbb{R}^\omega$ is given the product topology but not when given the box topology.

(3) A. Define both terms *connected* and *path connected*.
   B. Prove or give a counterexample: A path connected topological space is connected.
   C. Prove or give a counterexample: If $A \subset X$ is a path connected subset of a topological space, then its closure $\bar{A}$ is necessarily path connected.

(4) Prove that a closed subspace of a compact space is compact.

(5) Let $f\colon X \to \mathbb{R}$ be a smooth function on a smooth, compact, boundaryless manifold $X$. Suppose 0 is a regular value. Prove that $f^{-1}(0)$ is diffeomorphic to the boundary of some smooth compact manifold $W$.

(6) Let $X$ be a compact smooth manifold, $Y$ be smooth manifold, and $Z$ be a closed smooth submanifold of $Y$.
   A. Define what it means for the smooth function $f\colon X \to Y$ to be *transversal* to $Z$.
   B. Define what it means for two smooth functions $f_0, f_1\colon X \to Y$ to be *homotopic*.
   C. Show if $f_0, f_1\colon X \to Y$ are homotopic and both transversal to $Z$, then $I_2(f_0, Z) = I_2(f_1, Z)$. (That is, show their mod 2 intersection numbers with $Z$ are equal).

(7) Let $r = \sqrt{x^2 + y^2}$. For each non-negative integer $n$, determine whether the 1-form

$$\omega_n = \frac{-y\,dx + x\,dy}{r^n}$$

defined on $\mathbb{R}^2 - \{(0,0)\}$ is closed and whether it is exact.