TOPOLOGY PRELIMINARY EXAM

AUGUST 2013

1. Let X be a topological space. Prove that $A \subset X$ is open if and only if each point of A has an open neighborhood contained in A.

2. Prove that the following two statements about a subspace $A \subset \mathbb{R}^n$ are equivalent :

(a) A is compact,

(b) every continuous function $f : A \to \mathbb{R}$ attains its maximum on A.

3. Let $A_1, A_2, \ldots$ be a sequence of connected subspaces of a topological space X such that $A_n \cap A_{n+1}$ is non-empty for all n. Prove that the union $\bigcup A_n$ is connected.

4. Recall that ℓ^2 is the metric space of of all infinite sequences $\mathbf{x} = (x_1, x_2, \ldots)$ of real numbers such that the series $\sum x_n^2$ converges, with the metric

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum (x_n - y_n)^2\right)^{1/2}.$$

Let $A \subset \ell^2$ consist of all sequences $\mathbf{x}$ that are eventually zero, meaning that $x_n = 0$ for all but finitely many n.

(a) Prove that A is dense in ℓ^2.

(b) Use part (a) to prove that ℓ^2 is a separable topological space.

5. Prove that $\mathbb{R}^k$ and $\mathbb{R}^\ell$ are not diffeomorphic unless $k = \ell$.

6. Prove that the 2-sphere S^2 does not retract onto its equator $S^1 \subset S^2$, that is, there is no smooth map $r : S^2 \to S^1$ whose restriction to S^1 is the identity.

7. Let $\omega = (3 + 2xy)\,dx + (x^2 - 3y^2)\,dy$ be a 1-form on the xy–plane.

(a) Is ω closed ?

(b) Is ω exact ?

8. Ler $f_0, f_1 : X \to Y$ be homotopic maps between compact oriented manifolds of dimension n. Prove that for all closed n-forms ω on X, we have

$$\int_X f_0^* \omega = \int_X f_1^* \omega.$$