TOPOLOGY PRELIMINARY EXAM

AUGUST 2017

This exam consists of two parts, General Topology (this page) and Differential Topology (on the back). Each of the two parts contains five problems, and to pass that part, you need to solve four problems of your choice out of the five. Please state clearly which four problems you are solving.

1. GENERAL TOPOLOGY

1. Let X and Y be sequentially compact topological spaces. Prove that the space $X \times Y$ with the product topology is sequentially compact.

2. Prove that $\mathbb{R}_\ell$ does not have a countable basis of topology.

3. Let X and Y be topological spaces and assume that Y is Hausdorff. Given two continuous maps $f, g : X \to Y$ such that $f = g$ on a subset $A \subset X$ which is dense in X, prove that $f = g$ on the entire X.

4. Let X be a topological space which is connected, normal, and has at least two distinct points. Prove that there is a continuous surjective map $f : X \to [0, 1]$, assuming the topology on $[0, 1] \subset \mathbb{R}$ is induced by the standard topology on $\mathbb{R}$.

5. Let (X, d) be a complete metric space.

(a) Prove that if X is totally bounded then it is sequentially compact.

(b) Will the conclusion of (a) necessarily hold if X is bounded? Justify your answer.

2. DIFFERENTIAL TOPOLOGY

Problem 1. Prove that $S = \{ (x, y, z) \mid x^3 - y^3 + xyz - xy = 1 \} \subset \mathbb{R}^3$ is a smooth manifold. Find the tangent space to S at $(1, 1, 2)$.

Problem 2. Prove or disprove: the vector space V of real symmetric $n \times n$ matrices ($A^t = A$) with zero trace ($\operatorname{tr} A = 0$) and the vector space W of real skew-symmetric $n \times n$ matrices ($A^t = -A$) intersect transversely inside the vector space of all real $n \times n$ matrices.

Problem 3. Let $f, g : S^2 \to S^2$ be smooth maps of the unit 2-sphere $S^2 \subset \mathbb{R}^3$ such that $\deg f \neq \deg g$. Prove that there exists a point $x \in S^2$ such that $f(x) = -g(x)$.

Problem 4. Let M be a smooth manifold, $f : M \to \mathbb{R}$ a nowhere zero smooth function, and ω a differential 1-form on M such that $d(f\omega) = 0$. Prove that $w \wedge d\omega = 0$.

Problem 5. Use Stokes' Theorem to evaluate the line integral

$$\oint_C 3x^2 y \, dx + x^3 \, dy + 3(y^2 - x^2) \, dz,$$

where C is the curve of intersection of the surfaces $z = xy$ and $x^2 + y^2 = 1$ in $\mathbb{R}^3$, oriented counterclockwise as viewed from the top of the z-axis.