MS TOPOLOGY QUALIFYING EXAM

JUNE 2007

1. Let $f : X \to Y$ be a continuous bijective map from a compact topological space X onto a Hausdorff topological space Y. Show that f is a homeomorphism.

2. A subspace $A \subset \mathbb{R}^n$ has the property that every continuous function $f : A \to \mathbb{R}$ attains its maximum on A. Show that then A must be compact.

3. Let Z be a topological space and $Y \subset Z$ a retract of Z, meaning that there is a continuous map $r : Z \to Y$ such that $r(y) = y$ for every $y \in Y$. Show that if Z is Hausdorff then Y is closed in Z.

4. Show that for every $n \geq 3$ there is a continuous surjective map $g : I \to I^n$ where I is the closed interval [0,1]. Show also that such a map cannot be smooth.

5. Show that a smooth manifold X is path connected if and only if it is connected.

6. Let $f : X \to Y$ be a smooth map of compact smooth manifolds. Show that the map $F : X \to X \times Y$ given by $F(x) = (x, f(x))$ is an embedding.

7. Let $S^m \subset \mathbb{R}^{m+1}$ be the unit sphere $\|x\| = 1$. Prove that the antipodal map $\tau : S^{2n+1} \to S^{2n+1}$, $\tau(x) = -x$, is homotopic to the identity map. Is the same true for the antipodal map $\tau : S^{2n} \to S^{2n}$, $n \geq 1$? Why or why not ?

8. Let C be a smoothly embedded circle in the xy–plane not passing through the orinin. Evaluate

$$\int_C \frac{(2x^3 + 2xy^2 - 2y)\, dx + (2y^3 + 2x^2y + 2x)\, dy}{x^2 + y^2}$$

(make sure your answer covers all possible cases).