MS TOPOLOGY QUALIFYING EXAM

JUNE 2008

1. Let X be a topological space. Prove that a subspace $A \subset X$ is dense if and only if its complement $X - A$ has empty interior.

2. Prove that every continuous map $f : [0,1] \to [0,1]$ has a fixed point. Is the same true for continuous maps $(0,1) \to (0,1)$?

3. Prove that every compact Hausdorff space X is regular.

4. Recall that the metric space $\ell^2(\mathbb{R})$ consists of all sequences $\mathbf{x} = \{x_n\}$, $n \in \mathbb{Z}_+$, of real numbers such that $\sum x_n^2$ converges, with metric

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2}.$$

Prove that the unit ball $B_1(\mathbf{0}) \subset \ell^2(\mathbb{R})$ given by

$$\sum_{n=1}^{\infty} x_n^2 \leq 1$$

is not compact.

5. Let $f : X \to Y$ be a diffeomorphism of smooth manifolds. Show that, for every $x \in X$, the derivative $df_x : T_x X \to T_{f(x)} Y$ is an isomorphism.

6. Prove that the set X of all non–zero 2×2 real matrices A such that AA^t is diagonal (that is, its off diagonal entries are zero) is a submanifold of $\mathbb{R}^4 = M_{2\times 2}(\mathbb{R})$. What is the dimension of X ? Is X compact ?

7. Prove that there is no non–vanishing vector field on S^2.

8. Let $S \subset \mathbb{R}^3$ be the paraboloid $z = x^2 + y^2, \quad x^2 + y^2 \leq 1,$ oriented by the upward normal vector, and $\omega = dx \wedge (dy + dz)$. Evaluate

$$\int_S \omega.$$