MS TOPOLOGY QUALIFYING EXAM

JUNE 2009

(1) Let A be subset of a topological space X. Prove that A is open in X if and only if every point in A has a neighborhood contained in A.

(2) Let $f\colon X \to Y$ be a continuous bijective map from a compact topological space X to a Hausdorff topological space Y. Show that f is a homeomorphism.

(3) Prove that ever compact Hausdorff space X is regular.

(4) Let (X, d) be a metric space, $A \subset X$, and $x \in X$. Prove that the point x is in the closure of A if and only if there is a sequence of points $x_n \in A$ that converges to x.

(5) Let $S^m \subset \mathbb{R}^{m+1}$ be the unit sphere, $||x|| = 1$. The antipodal map $\alpha\colon S^m \to S^m$ is defined by $\alpha(x) = -x$. Prove if m is odd, then the antipodal map is homotopic to the identity map. Is the same true if m is even? Prove or disprove.

(6) Let $H \subset \mathbb{R}^3$ be the hyperboloid defined by $x^2 + y^2 - z^2 = 1$. For $a > 0$, let $S_a \subset \mathbb{R}^3$ be the sphere defined by $(x-2)^2 + y^2 + z^2 = a$. For what values of a does S_a intersect H transversally? For each value of a where S_a and H do not intersect transversally, what does the intersection look like?

(7) Let $f\colon X \to Y$ be a smooth map of compact smooth manifolds. Show that the map $F\colon X \to X \times Y$ given by $F(x) = (x, f(x))$ is an embedding.

(8) Show that the 1–form
$$\omega = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$$
on $\mathbb{R}^2 - \{(0,0)\}$ is closed but not exact.