MS TOPOLOGY QUALIFYING EXAM

JUNE 2010

1. Let X and Y be metric spaces, and $f, g : X \to Y$ continuous maps. Suppose that $f = g$ on a subset $A \subset X$ which is dense in X. Prove that then $f = g$ on the entire X.

2. Recall that a subspace A of a topological space X is called a *retract* of X if there is a continuous map $r : X \to A$ such that $r \circ i = 1$, where $i : A \to X$ is the inclusion map. Prove that $\{0, 1\}$ is not a retract of $[0, 1]$.

3. Let X be a Hausdorff topological space, and A, $B \subset X$ its compact subspaces. Prove that $A \cap B$ is compact.

4. Let (X, d) be a metric space. Recall that a map $f : X \to X$ is called a *contraction* of X if there is a number $c < 1$ such that

$$d(f(x), f(y)) \ \leq \ c \, d(x, y)$$

for all $x, y \in X$. Prove that if f is a contraction of a complete metric space X then there is a unique point $x \in X$ such that $f(x) = x$.

5. Let $f : X \to Y$ be a smooth map of compact manifolds of the same dimension, and $q \in Y$ a regular value of f. Prove that $f^{-1}(q)$ is finite.

6. Prove that all contractible manifolds are simply connected. Give an example which shows that the converse is not true.

7. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ given by $|x| = 1$. Compute the degree of the antipodal map $f : S^n \to S^n$ defined as $f(x) = -x$.

8. Use Stokes' Theorem to evaluate the integral

$$\oint_C 3x^2 y \, dx + x^3 \, dy + 3xy \, dz,$$

where C is the curve of intersection of the saddle $z = x^2 - y^2$ with the cylinder $x^2 + y^2 = 1$ oriented clockwise as viewed from the top of the z-axis.