MS TOPOLOGY PRELIMINARY EXAM

JUNE 2011

(1) Let X be a topological space. Prove that a subspace $A \subset X$ is dense if and only if its complement has empty interior.

(2) Let (X, d) be a compact metric space with continuous contraction $f\colon X \to X$. (The function f is a *contraction* if there exists a number $\alpha < 1$ such that $d(f(x), f(y)) < \alpha d(x, y)$ for any distinct points $x, y \in X$.)
Show there exists a unique point $x \in X$ such that $f(x) = x$.

(3) Let $f\colon X \to Y$ be a function between topological spaces. Prove f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for any subset A in X. (Here, $\overline{S}$ denotes the closure of the set S.)

(4) Prove the Intermediate Value Theorem:

> Let $f\colon X \to Y$ be a continuous map of the connected space X to the ordered set Y in the order topology. If a and b are two points of X and r is a point of Y between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

(5) Prove the graph of $f(x, y) = \sqrt{x^2 + y^2}$ is homeomorphic but not diffeomorphic to $\mathbb{R}^2$.

(6) For which values of $r > 0$ does $H = \{x^2 + y^2 - z^2 - w^2 = 1\} \subset \mathbb{R}^4$ transversally intersect $S^3_r \subset \mathbb{R}^4$, the 3-sphere of radius r centered at the origin?

(7) Let X be a smooth, compact, boundaryless n-manifold and $f\colon X \to \mathbb{R}^n$ a smooth map. Prove that f must have a critical point.

(8) Consider the 2-form $\omega = z\, dx \wedge dy$ on $\mathbb{R}^3$.
  (a) Is ω exact?
  (b) Let M be the paraboloid $z - x^2 - y^2 = 1$ in $\mathbb{R}^3$. Is the restriction of ω to M exact?