TOPOLOGY PRELIMINARY EXAM

JUNE 2013

1. Let X and Y be topological spaces and $f : X \to Y$ a map which is continuous and bijective.

(a) Give an example showing that f need not be a homeomorphism. Be sure to prove that the map f in your example is continuous and that it is not a homeomorphism.

(b) Prove that f must be a homeomorphism under the additional assumption that X is compact and Y is Hausdorff.

2. Let X be a non-empty compact topological space.

(a) Give an outline of the proof of the extreme value theorem: for any continuous map $f : X \to \mathbb{R}$ its image $f(X)$ has a maximum element.

(b) Part (a) presumes that $\mathbb{R}$ has the Euclidean topology. Is the statement also true if $\mathbb{R}$ has the finite complement topology?

3. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Prove that the sequence of real numbers $\{d(x_n, y_n)\}$ converges.

4. Prove that the union of the two coordinate axes in $\mathbb{R}^2$ is not a manifold.

5. The special orthogonal group $SO(n)$, $n \geq 1$, consists of all real $n \times n$ matrices A such that $AA^t = E$ and $\det A = 1$. Prove that $SO(n)$ is a smooth manifold. What is its dimension?

6. Prove that there is no such manifold Y that the sphere S^n of dimension $n \geq 2$ is diffeomorphic to $S^1 \times Y$.

7. Let S^n, $n \geq 1$, be the unit sphere in $\mathbb{R}^{n+1}$ given by $\|x\| = 1$. Suppose $f : S^n \to S^n$ is a smooth map such that the position vector of $x \in S^n$ and the position vector of $f(x) \in S^n$ are perpendicular to each other for all $x \in S^n$. Prove that f is homotopic to the identity map of S^n. Then show that such an f exists if and only if n is odd.

8. Let $X \subset \mathbb{R}^3$ be a compact connected orientable surface without boundary, and let $\omega \in \Omega^1(X)$. Prove that $d\omega$ must vanish at some point of X.