TOPOLOGY PRELIMINARY EXAM

JUNE 2021

1. Let $\mathbb{R}^\omega$ be the Cartesian product of countably many copies of the real line, and $f : \mathbb{R} \to \mathbb{R}^\omega$ the diagonal function $f(x) = (x, x, \ldots, x, \ldots)$.

(a) Is f continuous with respect to the product topology on $\mathbb{R}^\omega$? Explain.

(b) Is f continuous with respect to the box topology on $\mathbb{R}^\omega$? Explain.

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions defined by the formula

$$f_n(x) = \begin{cases} \dfrac{1}{n}\sqrt{n^2 - x^2}, & \text{if } |x| < n, \\ 0, & \text{if } |x| \geq n. \end{cases}$$

(a) Show that the sequence $f_n(x)$ converges pointwise and find its limit.

(b) Does the sequence $f_n(x)$ converge uniformly? Explain.

3. Prove or disprove the following statements with brief but complete reasoning:

(a) Any infinite set with the finite complement topology is connected.

(b) The real line with the lower limit topology is connected.

(c) The complement of the central circle in the Möbius band is connected.

(c) The union of the graphs $y = e^x$ and $y = 0$ in the coordinate xy–plane is connected.

4. Let X be a Hausdorff topological space and suppose that you are given a nested collection $\ldots \subset C_n \subset C_{n-1} \subset \ldots \subset C_2 \subset C_1$ of non-empty compact subspaces of X. Prove that the intersection

$$C = \bigcap_{n=1}^{\infty} C_n$$

is non-empty and compact.

5. Prove that the set $X = \{ (x, y) \in \mathbb{R}^2 \mid x^2 = y^3 \}$ is not a smooth manifold.

6. Let S^1 be the unit circle $x^2 + y^2 = 1$ in the coordinate xy–plane. Find all the critical points of the function $f : S^1 \to \mathbb{R}$ given by the formula $f(x, y) = xy$.

7. Let X be a compact smooth oriented manifold with boundary. Prove that there does not exist a smooth retraction of X onto its boundary ∂X.

8. Let the Euclidean space $\mathbb{R}^4$ have coordinates x, y, z, and w. Integrate the differential 2-form

$$dz \wedge dw + xz \, dy \wedge dw$$

over the torus $T \subset \mathbb{R}^4$ given by the equations $x^2 + y^2 = 1$ and $z^2 + w^2 = 1$.