PREPARE FOR SUCCESS IN MTH107 & MTH105

Welcome 'Canes!

If you are currently enrolled in MTH107 or MTH105 for the upcoming semester, we encourage you to spend some time on this packet before the start of the term. We have carefully selected topics that are essential for success in this course.

It is important that you work through this packet on your own, so that you get a clear gauge of your preparation. For each main topic, you will find examples worked out with explanations. After the examples, we have selected problems for you to try on your own. The answers to these problems are found at the end, so you can check your work.

If you feel you need to review certain topics in more depth, we recommend these websites – just search the related topics on the site:

https://www.khanacademy.org/math/algebra-home

YouTube - Patrick JMT

Here's to a great start of the semester and much SUCCESS!!

MTH107/105 Prerequisite Review

TOPICS

Factoring polynomials

Greatest Common Factor

Examples. Factor out the greatest common factor and simplify where possible.

- $100m^5 - 50m^4 + 25m^3 = 25m^3\left(4m^2 - 2m + 1\right)$
- $5x^4y^3 + 15x^5y^4 - 20x^6y^7 = 5x^4y^3\left(1 + 3xy - 4x^2y^4\right)$
- $2(5-x)^3 - 3(5-x)^2$

  $= (5-x)^2\left[2(5-x) - 3\right] = (5-x)^2(10 - 2x - 3) = (5-x)^2(-2x+7)$ *or* $-(5-x)^2(2x-7)$

- Factoring out a negative exponent: $-2q^{-3} + 8q^{-2}$

Recall, when we factor out a GCF, we are *dividing* each term by that GCF (divide the coefficients and subtract the exponents). Here, the smallest exponent on the q is –3 (–3 is smaller than –2). We will also factor out a GCF of –2 instead of +2. I will write these as fractions and simplify them as a reminder of what is happening:

$$-2q^{-3}\left(\frac{-2q^{-3}}{-2q^{-3}} + \frac{8q^{-2}}{-2q^{-3}}\right) = -2q^{-3}\left(\frac{\cancel{-2q^{-3}}}{\cancel{-2q^{-3}}} + \frac{\overset{-4}{\cancel{8}}\,q^{-2-(-3)}}{\cancel{-2}\,\cancel{q^{-3}}}\right) = -2q^{-3}\left(1 + \frac{-4q^{-2+3}}{1}\right) = -2q^{-3}(1-4q) = \frac{-2(1-4q)}{q^3}$$

Factoring by Grouping

Examples. Factor Completely

- $10x^2y^2 - 18 + 15y^2 - 12x^2$

  Notice, if we group the first and last two terms, we do not end up with a GCF that can be factored out. Let's rearrange the terms so that at least two of them have variables and then proceed with grouping:

  $= 10x^2y^2 + 15y^2 - 18 - 12x^2$

  $= \left(10x^2y^2 + 15y^2\right) + \left(-18 - 12x^2\right)$

  $= 5y^2\left(2x^2 + 3\right) + (-6)\left(3 + 2x^2\right)$

  $= \left(2x^2 + 3\right)\left(5y^2 - 6\right)$

- $x^3 + 4x - 2x^2 - 8$

  $= \left(x^3 + 4x\right) + \left(-2x^2 - 8\right)$
  $= x\left(x^2 + 4\right) + \left(-2\right)\left(x^2 + 4\right)$
  $= \left(x^2 + 4\right)\left(x - 2\right)$

- $6x^3y^2 + 3x^2y^2 - 42xy^2 - 21y^2$

  Notice that we have a GCF. Factor out the GCF *first* and *then* proceed with grouping:

  $= 3y^2\left(2x^3 + x^2 - 14x - 7\right)$
  $= 3y^2\left[\left(2x^3 + x^2\right) + \left(-14x - 7\right)\right]$
  $= 3y^2\left[x^2\left(2x+1\right) + \left(-7\right)\left(2x+1\right)\right]$
  $= 3y^2\left(2x+1\right)\left(x^2 - 7\right)$

<u>Factoring Trinomials</u>

Trinomials with leading coefficient of 1: $x^2 + bx + c$

Examples. Factor Completely.

- $x^2 + 9x + 20 = \left(x + 4\right)\left(x + 5\right)$
- $a^2 - 7a + 10 = \left(a - 5\right)\left(a - 2\right)$
- $x^2 + 3x - 5$ This trinomial is prime and not factorable over the integers.
- $t^2 - 5tp + 6p^2 = \left(t - 3p\right)\left(t - 2p\right)$

Trinomials of the form $ax^2 + bx + c$, *where* $a \neq 1$

Factor completely. NOTE: you may be used to using the "AC method," which works like grouping. I will demonstrate this method once. However, you are not in basic algebra anymore, you are in precalculus, and you need to be proficient at factoring these by inspection *without* using the AC method.

The steps for the AC method were:

1) Multiply a and c.
2) Find all pairs of numbers whose product is ac.
3) Find the pair which combines (either addition or subtract as needed) to make b.
4) Rewrite the trinomial showing this combination, which creates a polynomial with 4 terms.
5) Factor by grouping.

- $6k^2 - 19k + 10$

  $ac = (6)(10) = 60$

The pairs of factors are: 1*60   2*30   3*10   4*15   5*12   6*10

We know that because our constant is positive, the factors must be the *same* sign and the numbers must *add* to –19. Therefore, we must use 4 and 15, but they must be negative. Our trinomial becomes:

$$6k^2 - 4k - 15k + 10$$

Proceed to factor by grouping:

$$= (6k^2 - 4k) + (-15k + 10)$$
$$= 2k(3k - 2) + (-5)(3k - 2)$$
$$= (3k - 2)(2k - 5)$$

This example demonstrates why the AC method is not ideal – when the product has many factors this is very time-consuming. Practice factoring these polynomials WITHOUT using the AC method based on your understanding of how *multiplying* binomials works!

- $6x^2 + 13x - 5 = (2x + 5)(3x - 1)$

- $2m^3 - 4m^2 - 6m$

  $= 2m(m^2 - 2m - 3)$

  $= 2m(m - 3)(m + 1)$

- $8(x+5)^2 - 2(x+5) - 3$

This trinomial is *quadratic in form*. Instead of just an x we have $x + 5$. I will rewrite this using a different variable. Let's replace $x + 5$ with the letter u:

$$8u^2 - 2u - 3$$

Now, factor.

$$= (2u+1)(4u-3)$$

Remember that we did not start with u, so we need to put our $x + 5$ back in, using parentheses and simplify each factor when possible:

$$= (2(x+5)+1)(4(x+5)-3)$$
$$= (2x+10+1)(4x+20-3)$$
$$= (2x+11)(4x+17)$$

- $6r^4 - 13r^2 - 5$

This trinomial is also quadratic in form.

$$6r^4 - 13r^2 - 5$$
$$= 6(r^2)^2 - 13r^2 - 5$$

Let's replace r^2 with the letter u:

$$= 6u^2 - 13u - 5$$
$$= (2u-5)(3u+1)$$

Replacing the u we have:

$$(2r^2 - 5)(3r^2 + 1)$$

**You should practice factoring these *without* having to make the u-substitution.

Special Factoring – Examples

Difference of Squares: $A^2 - B^2 = (A+B)(A-B)$

- $144y^2 - 4z^2$

  $= 4\left(36y^2 - z^2\right)$
  $= 4\left(\left(6y\right)^2 - z^2\right)$
  $= 4\left(6y+z\right)\left(6y-z\right)$

- $x^4 - 16$

  $= \left(x^2\right)^2 - \left(4\right)^2 = \left(x^2+4\right)\left(x^2-4\right)$

  Notice, we are not finished as one of the factors is another difference of squares:

  $= \left(x^2+4\right)\left(x^2-4\right) = \left(x^2+4\right)\left(x-2\right)\left(x+2\right)$

Perfect Square:
$A^2 + 2AB + B^2 = (A+B)^2$
$A^2 - 2AB + B^2 = (A-B)^2$

- $9a^2 + 48ab + 64b^2 = \left(3a\right)^2 + 2\left(3a\right)\left(8b\right) + \left(8b\right)^2 = \left(3a+8b\right)^2$

- $49x^2 - 14x + 1 = \left(7x\right)^2 - 2\left(7x\right)\left(1\right) + \left(1\right)^2 = \left(7x-1\right)^2$

Difference of Cubes: $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

- $27m^3 - 64 = \left(3m\right)^3 - \left(4\right)^3 = \left(3m-4\right)\left(\left(3m\right)^2 + \left(3m\right)\left(4\right) + \left(4\right)^2\right) = \left(3m-4\right)\left(9m^2 + 12m + 16\right)$

Sum of Cubes: $A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

- $125x^3 + 27y^3$

$= \left(5x\right)^3 + \left(3y\right)^3 = \left(5x+3y\right)\left(\left(5x\right)^2 - \left(5x\right)\left(3y\right) + \left(3y\right)^2\right) = \left(5x+3y\right)\left(25x^2 - 15xy + 9y^2\right)$

More Examples: Mixed exercises. Factor completely.

- $225p^2 - 256$

$= (15p)^2 - (16)^2 = (15p+16)(15p-16)$

- $18x^3y + 3x^2y^2 - 6xy^3$

$= 3xy\left(6x^2 + xy - 2y^2\right) = 3xy(2x-y)(3x+2y)$

- $16x^3 + 32x^2 - 9x - 18$

$= \left(16x^3 + 32x^2\right) + (-9x - 18)$
$= 16x^2(x+2) - 9(x+2)$
$= (x+2)\left(16x^2 - 9\right)$
$= (x+2)(4x+3)(4x-3)$

- $x^5 - 8x^2 - 4x^3 + 32$

$= \left(x^5 - 8x^2\right) + \left(-4x^3 + 32\right)$
$= x^2\left(x^3 - 8\right) - 4\left(x^3 - 8\right)$
$= \left(x^3 - 8\right)\left(x^2 - 4\right)$
$= (x-2)\left(x^2 + 2x + 4\right)(x+2)(x-2)$

- $(5r+2s)^2 - 6(5r+2s) + 9$

Let $u = 5r + 2s$: $u^2 - 6u + 9 = (u-3)^2$

Replace u: $= (5r + 2s - 3)^2$

**NOTE: It is possible to factor $x^2 - 5$ using something *other* than integers. Notice:

$\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right) = x^2 - \sqrt{5}x + \sqrt{5}x - \left(\sqrt{5}\right)^2 = x^2 - 5$

We will return to this idea later.

Your Turn. Factor completely.

1. $20+5m+12n+3mn$

2. $y^2+7y-30$

3. $20x^2+47x+24$

4. $25x^2-90x+81$

5. $24x^2+42x+15$

6. p^4-10p^2+16

7. x^4-81

8. x^5+3x^4-x-3

9. $2a^3+a^2-14a-7$

10. $p^2+15p+56$

11. $27z^2+42z-5$

12. $14c^2-17cd-6d^2$

13. $6a^3+12a^2-90a$

14. $100a^2-9b^2$

15. $16x^3+32x^2-9x-18$

16. x^9-1

Simplifying expressions

Properties of Exponents and Rational Expressions

Examples. Simplify the following. Assume all variables are non-zero and express your answer using only positive exponents.

- $\left(-5x^2y^4\right)\left(\frac{2}{5}xy\right) = -2x^3y^5$

- $\left(\frac{27x^{-3}y^6}{18x^{-4}y^{-5}}\right)^{-2} = \left(\frac{3x^{-3+4}y^{6+5}}{2}\right)^{-2} = \left(\frac{3xy^{11}}{2}\right)^{-2} = \left(\frac{2}{3xy^{11}}\right)^2 = \frac{4}{9x^2y^{22}}$

- $\left(\frac{x^{-2/3}}{y^{-3/4}}\right)^4\left(\frac{x^{-3/8}}{y^{-1/4}}\right)^{-2} = \left(\frac{x^{-2/3}}{y^{-3/4}}\right)^4\left(\frac{x^{-3/8}}{y^{-1/4}}\right)^{-2} = \frac{x^{-8/3}\cdot x^{3/4}}{y^{-3}\cdot y^{1/2}} = \frac{x^{-15/12}}{y^{-5/2}} = \frac{y^{5/2}}{x^{5/4}}$

- $\frac{x^2-25}{x^2+x-20} \div \frac{x^2-2x-15}{x^2+7x+12}$

  $= \frac{(x+5)(x-5)}{(x+5)(x-4)} \div \frac{(x-5)(x+3)}{(x+4)(x+3)} = \frac{\cancel{(x+5)}\,\cancel{(x-5)}}{\cancel{(x+5)}\,(x-4)} \cdot \frac{(x+4)\,\cancel{(x+3)}}{\cancel{(x-5)}\,\cancel{(x+3)}} = \frac{x+4}{x-4}$

- $\frac{x-3}{x+2} - \frac{x+4}{x-2}$

  $= \frac{(x-3)(x-2)-(x+4)(x+2)}{(x+2)(x-2)} = \frac{x^2-5x+6-x^2-6x-8}{(x+2)(x-2)} = \frac{-11x-2}{(x+2)(x-2)}$

Complex Fractions (Mixed Quotients)

Recall that to divide rational expressions you must multiply by the reciprocal of the denominator. This only works if you are dividing a single rational expression by another.

There are two methods for simplifying complex fractions.

Method 1:

$$\frac{\frac{2}{x}-\frac{3}{y}}{\frac{4}{x^2}+\frac{6}{y^2}} = \frac{\frac{2y}{xy}-\frac{3x}{yx}}{\frac{4y^2}{x^2y^2}+\frac{6x^2}{x^2y^2}} = \frac{\frac{2y-3x}{xy}}{\frac{4y^2+6x^2}{x^2y^2}} = \frac{2y-3x}{xy}\cdot\frac{x^2y^2}{4y^2+6x^2} = \frac{xy(2y-3x)}{4y^2+6x^2}$$

Method 2:

$$\frac{\frac{2}{x}-\frac{3}{y}}{\frac{4}{x^2}+\frac{6}{y^2}} = \left(\frac{\frac{2}{x}-\frac{3}{y}}{\frac{4}{x^2}+\frac{6}{y^2}}\right)\cdot\frac{x^2y^2}{x^2y^2} = \frac{\frac{2x^2y^2}{x}-\frac{3x^2y^2}{y}}{\frac{4x^2y^2}{x^2}+\frac{6x^2y^2}{y^2}} = \frac{2xy^2-3x^2y}{4y^2+6x^2} = \frac{xy(2y-3x)}{4y^2+6x^2}$$

Examples. Simplify the expression. Write the expression as a single quotient in which only positive exponents and/or radicals appear.

- $$\frac{\frac{2}{x}-\frac{3}{y}}{\frac{4y}{x}-\frac{9x}{y}} = \frac{\left(\frac{2}{x}-\frac{3}{y}\right)}{\left(\frac{4y}{x}-\frac{9x}{y}\right)}\cdot\frac{(xy)}{(xy)} = \frac{2y-3x}{4y^2-9x^2} = \frac{2y-3x}{(2y+3x)(2y-3x)} = \frac{1}{2y+3x}$$

- $$\frac{x^{-2}-25y^{-2}}{x^{-1}-5y^{-1}}$$

CAUTION: $\frac{1}{a+b}=(a+b)^{-1} \neq a^{-1}+b^{-1}$

$$= \frac{\left(\frac{1}{x^2}-\frac{25}{y^2}\right)}{\left(\frac{1}{x}-\frac{5}{y}\right)}\cdot\frac{x^2y^2}{x^2y^2} = \frac{y^2-25x^2}{xy^2-5x^2y} = \frac{(y+5x)(y-5x)}{xy(y-5x)} = \frac{y+5x}{xy}$$

- $$\frac{\sqrt{x+1}-\frac{x+8}{2\sqrt{x+1}}}{x+1} = \frac{\left(\sqrt{x+1}-\frac{x+8}{2\sqrt{x+1}}\right)}{(x+1)}\cdot\frac{2\sqrt{x+1}}{2\sqrt{x+1}} = \frac{2(x+1)-(x+8)}{2(x+1)^{3/2}} = \frac{x-6}{2(x+1)^{3/2}}$$

Your Turn. Perform the indicated operation. Simplify if possible.

17. $\left(-3x^2y^4w^3\right)^2$

18. $\left(\dfrac{24x^{-2}y^{-5}}{12x^3y^3}\right)^3$

19. $\dfrac{\left(4x^3y\right)^{-2}\left(2w\right)^5}{\left(2xy\right)^3\left(6wy\right)^{-2}}$

20. $\dfrac{c^2+2c}{c^2-4}\cdot\dfrac{c^2-4c+4}{c^2-c}$

21. $\dfrac{z^2+2z}{5+z}\div\dfrac{4-z^2}{3z-6}$

22. $\dfrac{15}{x^2+3x}+\dfrac{2}{x}+\dfrac{5}{x+3}$

23. $\dfrac{4}{x+1}+\dfrac{1}{x^2-x+1}-\dfrac{12}{x^3+1}$

24. $\dfrac{\dfrac{x}{y}-\dfrac{16y}{x}}{\dfrac{1}{y}-\dfrac{4}{x}}$

25 $\dfrac{\dfrac{x}{x-2}+1}{\dfrac{3}{x^2-4}+1}$

26 $\dfrac{1-5x^{-1}-24x^{-2}}{1-15x^{-1}+56x^{-2}}$

Solving equations

Quadratic equations

Factoring.

- $y(y-9)=-14$

$y^2-9y+14=0 \;\rightarrow\; (y-7)(y-2)=0$

$y-7=0 \;\rightarrow\; y=7$

$y-2=0 \;\rightarrow\; y=2$

$\{2,\ 7\}$

The square root property.

- $3x^2-54=0$

$x^2=18 \;\rightarrow\; x=\pm\sqrt{18}=\pm 3\sqrt{2}$

$\{\pm 3\sqrt{2}\}$

- $(2x+3)^2=36$

$2x+3=\pm\sqrt{36} \;\rightarrow\; x=\dfrac{-3\pm 6}{2}$

$\left\{-\dfrac{9}{2},\ \dfrac{3}{2}\right\}$

Completing the square.

- $x^2+6x+2=0$

$x^2+6x+\left(\dfrac{6}{2}\right)^2=-2+\left(\dfrac{6}{2}\right)^2 \;\rightarrow\; x^2+6x+9=-2+9 \;\rightarrow\; (x+3)^2=7 \;\rightarrow\; x+3=\pm\sqrt{7} \;\rightarrow\; x=-3\pm\sqrt{7}$

$\{-3\pm\sqrt{7}\}$

- $x^2-3x-10=0$

$x^2-3x+\left(\dfrac{-3}{2}\right)^2=10+\left(\dfrac{-3}{2}\right)^2 \;\rightarrow\; x^2-3x+\dfrac{9}{4}=10+\dfrac{9}{4} \;\rightarrow\; \left(x-\dfrac{3}{2}\right)^2=\dfrac{49}{4} \;\rightarrow\; x-\dfrac{3}{2}=\pm\sqrt{\dfrac{49}{4}} \;\rightarrow\; x=\dfrac{3}{2}\pm\dfrac{7}{2}$

$\{-2,\ 5\}$

***NOTE:** Completing the square is “nice” and fast when the leading coefficient is 1 and the middle coefficient is an even number!

The quadratic formula.

- $5x^2 - 3x + 4 = 0$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)(4)}}{2(5)} = \frac{9 \pm \sqrt{-71}}{10} = \frac{9 \pm i\sqrt{71}}{10}$$

$$\left\{ \frac{9 \pm i\sqrt{71}}{10} \right\}$$

- $-2x(x+2) = -3$

$$-2x^2 - 4x + 3 = 0 \rightarrow 2x^2 + 4x - 3 = 0 \rightarrow x = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(-3)}}{2(2)} = \frac{-4 \pm \sqrt{40}}{4} = \frac{-4 \pm 2\sqrt{10}}{4} = \frac{-2 \pm \sqrt{10}}{2}$$

$$\left\{ \frac{-2 \pm \sqrt{10}}{2} \right\}$$

***NOTE:** The quadratic formula works for any quadratic, but it is not always the easiest or fastest method!

Rational

- $3x = 1 - \frac{1}{x}$

$$3x^2 = x - 1 \text{ and } x \neq \{0\} \rightarrow 3x^2 - x + 1 = 0 \rightarrow x = \frac{1 \pm \sqrt{1-12}}{6}$$

$$\left\{ \frac{1 \pm i\sqrt{11}}{6} \right\}$$

- $\frac{1}{x-5} = \frac{1}{x^2 - 4x - 5} + \frac{6}{x+1}$

$$\left(\frac{1}{x-5}\right)(x-5)(x+1) = \left[\frac{1}{(x-5)(x+1)} + \frac{6}{x+1}\right](x-5)(x+1) \quad \text{and } x \neq \{-1, 5\}$$

$$x + 1 = 1 + 6(x-5) \rightarrow x + 1 = 1 + 6x - 30 \rightarrow -5x = -30 \rightarrow x = 6$$

$$\{6\}$$

Factoring

- $p^3 - 25p = 0$

$$p(p+5)(p-5) = 0 \rightarrow p = \{0, \pm 5\}$$

- $x^4 + 3x^3 = 4x^2 + 12x$

$x^4 + 3x^3 - 4x^2 - 12x = 0 \;\rightarrow\; x\left(x^3 + 3x^2 - 4x - 12\right) = 0$

$x\left(x^2(x+3) - 4(x+3)\right) = 0 \;\rightarrow\; x(x+3)\left(x^2 - 4\right) = 0 \;\rightarrow\; x = \{0, -3, \pm 2\}$

- $x^3 + 8 = 0$

$x^3 + 2^3 = 0 \;\rightarrow\; (x+2)\left(x^2 - 2x + 4\right) = 0$

$x + 2 = 0 \rightarrow \; x = -2$

$x^2 - 2x + 4 \rightarrow \; x = \dfrac{2 \pm \sqrt{4-16}}{2} = \dfrac{2 \pm \sqrt{-12}}{2} = \dfrac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$

$\left\{-2,\ 1 \pm i\sqrt{3}\right\}$

Quadratic in form

- $2x^4 + 4 = 7x^2$

$2x^4 - 7x^2 + 4 = 0 \;\rightarrow\; \left(2x^2 + 1\right)\left(x^2 - 4\right) = 0$

$2x^2 + 1 \;\rightarrow\; x = \pm\sqrt{-\dfrac{1}{2}} = \pm\dfrac{i}{\sqrt{2}} = \pm\dfrac{i\sqrt{2}}{2}$

$x^2 - 4 = 0 \;\rightarrow\; x = \pm 2$

$\left\{\pm 2,\ \pm\dfrac{i\sqrt{2}}{2}\right\}$

- $8 + 36k^{-2} = 54k^{-1}$

$36k^{-2} - 54k^{-1} + 8 = 0 \;\rightarrow\; 18k^{-2} - 27k^{-1} + 4 = 0 \;\rightarrow\; \left(6k^{-1} - 1\right)\left(3k^{-1} - 4\right) = 0$

$6k^{-1} - 1 = 0 \;\rightarrow\; k^{-1} = \dfrac{1}{6} \;\rightarrow\; k = 6$

$3k^{-1} - 4 = 0 \;\rightarrow\; k^{-1} = \dfrac{4}{3} \;\rightarrow\; k = \dfrac{3}{4}$

$\left\{\dfrac{3}{4},\ 6\right\}$

- $x^{2/3} + 2x^{1/3} - 3 = 0$

$\left(x^{1/3} + 3\right)\left(x^{1/3} - 1\right) = 0$

$x^{1/3} + 3 = 0 \;\rightarrow\; x^{1/3} = -3 \;\rightarrow\; x = -27$

$x^{1/3} - 1 = 0 \;\rightarrow\; x^{1/3} = 1 \;\rightarrow\; x = 1$

$\{-27, 1\}$

Square root

- $\sqrt{8-t} - \sqrt{26+t} = -2$

$\left(\sqrt{8-t}\right)^2 = \left(-2 + \sqrt{26+t}\right)^2$

$8 - t = 4 - 4\sqrt{26+t} + 26 + t$

$-2t - 22 = -4\sqrt{26+t}$

$t + 11 = 2\sqrt{26+t}$

$t^2 + 22t + 121 = 4(26+t) \;\rightarrow t^2 + 18t + 17 = 0 \;\rightarrow\; (t+17)(t+1) = 0$

$\sqrt{8-(-17)} - \sqrt{26+(-17)} = 5 - 3 = 2 \neq -2$

$\sqrt{8-(-1)} - \sqrt{26+(-1)} = 3 - 5 = -2$

$\{-1\}$

***NOTE:** When you square both sides of an equation like this, you must ALWAYS check your answers!

Absolute Value

Recall the definition of absolute value: $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$

- $|2x-5| - 3 = 12$

$|2x-5| = 15$

$2x - 5 = \pm 15 \;\rightarrow\; x = \dfrac{5 \pm 15}{2} \;\rightarrow\; x = \{-5, 10\}$

Your Turn. Find all solutions to each equation.

Provide exact answers (no decimals). Simplify all answers as much as possible.

27. $\frac{-12}{x} = x + 8$

28. $\frac{3}{2x} - \frac{1}{4x+2} = 1$

29. $(4x-7)^2 = 81$

30. $2x^2 - 5x + 3 = 0$

31. $x^4 - 29x^2 + 100 = 0$

32. $(2x+5)^2 - 6 = 4(2x+5)$

33. $4m^{4/3} - 13m^{2/3} + 9 = 0$

34. $3x = \sqrt{16 - 10x}$

35. $(3x-5)^2 + 12 = 0$

36. $\frac{12}{g^2 - 2g} + 3 = \frac{6}{g-2}$

37. $(x+9)(x-1) = (x+1)^2$

38. $\sqrt{x-6} + 2 = 5$

Distance and midpoint

Suppose we are given two arbitrary points $(x_1,\, y_1)$ *and* $(x_2,\, y_2)$ and we want to find the distance between them: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Let the points $P_1 = (x_1,\, y_1)$ *and* $P_2 = (x_2,\, y_2)$ be the endpoints of a line segment. Let the point $M = (x, y)$ be the midpoint of the line segment (i.e. equidistant from the endpoints). Then, the coordinates of the midpoint M are given by $M = (x, y) = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Your Turn. Answer the questions.

39. Find the distance between the points (-3, 2) and (0, -4).

40. Find all points having a y-coordinate of -3 whose distance from the point (1, 2) is 13.

41. The *midpoint* of the line segment from P_1 to P_2 is $(-1, 3)$. If $P_1 = (-4, 5)$, what is P_2 ?

Equations of lines

A linear equation in two variables can be written in several ways:

Standard Form (aka general): $Ax + By = C$, where $A > 0$ and A, B, and C are integers.

Slope-Intercept Form: $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.

Point-Slope Form: $y - y_1 = m(x - x_1)$, where m is slope; $(x_1,\, y_1)$ is a point on the line.

The slope m of a line passing through distinct points $(x_1,\, y_1)$ and $(x_2,\, y_2)$ is given by:

$$m = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

*If $y_1 = y_2$, the line is horizontal with slope zero and if $x_1 = x_2$, the line is vertical with undefined slope.

Two lines are parallel if they have the same slope.

Two lines are perpendicular if their slopes are negative reciprocals.

Your Turn. Lines.

Find the slope, *x*- and *y*-intercepts, and graph the line.

42. $2x + 3y = 6$

43. $y = \frac{1}{4}x - 3$

44. Find the equation of a line passing through the points $(-4,\ 3)$ and $(2,\ -6)$. Write the equation in standard form.

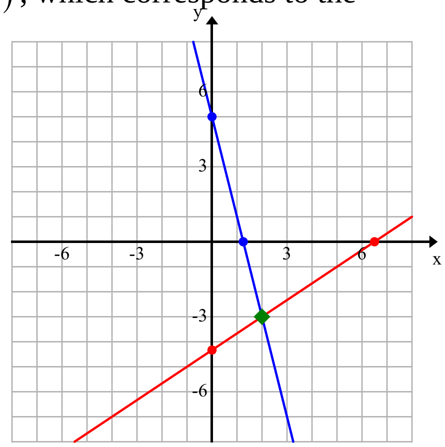
45. Find the equation of the line (in slope-intercept form) passing through the point (-4, 6) and

   a. Parallel to the line $2x - 3y = 8$

   b. Perpendicular to the line $3x + 5y = 9$

Systems of linear equations

Recall that the solution to a system of linear equations is an ordered pair (x, y), which corresponds to the point of intersection of the graphs of the two lines.



Example: $$\begin{aligned} 4x + y &= 5 \\ 2x - 3y &= 13 \end{aligned}$$

We can see from the graph that the two lines appear to intersect at the point $(2, -3)$. It is not always easy to determine the point of intersection simply by inspecting a graph. We find this solution by solving the system algebraically using one of two methods: substitution or elimination.

Method 1: Substitution. Select one equation and one variable in that equation. Solve for the variable. Then, substitute your expression into the *other* equation.

The variable *y* is easy to solve for in the first equation: $4x + y = 5 \rightarrow y = 5 - 4x$
Substitute this expression into the second equation: $2x - 3y = 13 \rightarrow 2x - 3(5 - 4x) = 13$
Solve this equation and we get $x = 2$.
We still need our *y*-coordinate. As we already have an equation that is solved for *y*, we plug in $x = 2$ and evaluate: $y = 5 - 4x \rightarrow y = 5 - 4(2) = -3$
Therefore, the solution to our system is the point $(2, -3)$.

Method 2: Elimination (aka Addition). The goal here is to manipulate one or both equations such that the coefficients on *one* of the variables is the same number, but opposite sign, in both equations. Then, we add the equations, term by term. If done correctly, one of the variables should be eliminated.

For our example, we can see that we can make the coefficients of the variable *x* both equal to 4 by multiplying the *second* equation by a factor of 2. However, remember that we want them to be opposite signs, so we multiply by –2:

$$(-2)(2x - 3y) = (13)(-2)$$

Now we have the following system and we add *vertically*, term by term:

$$\begin{aligned} 4x + y &= 5 \\ -4x + 6y &= -26 \\ \hline 7y &= -21 \rightarrow y = -3 \end{aligned}$$

Once again, the solution to our system is found to be the point $(2, -3)$.

Your turn. Solve the systems algebraically using both methods. Then, graph the lines and inspect the graphs to see if your answers seem *reasonable*.

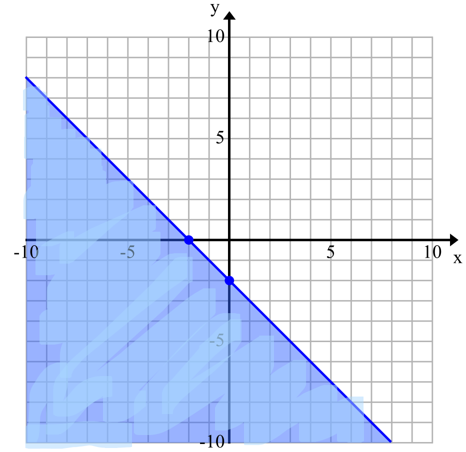
46. $\begin{cases} 2x + y = 5 \\ -x + 3y = 6 \end{cases}$

47. $\begin{cases} 2x + 3y = 19 \\ 3x - 7y = -6 \end{cases}$

Linear inequalities in two variables

Examples.

- $y < -x - 2$

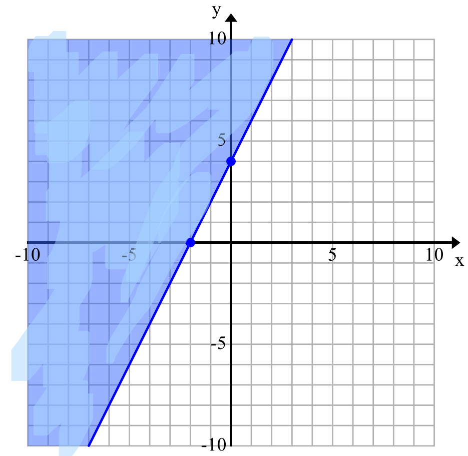
  

  Graph a dashed/dotted line for the equation $y = -x - 2$

  Slope: $m = -1$ x-int: $(-2, 0)$ y-int: $(0, -2)$

  Shade all y-values below the line.

- $y \geq 2x + 4$

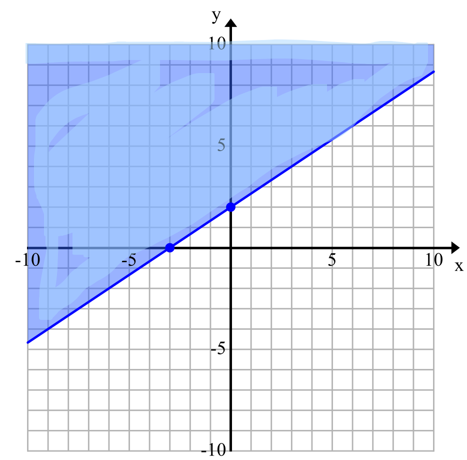
  

  Graph a solid line for the equation $y = 2x + 4$

  Slope: $m = 2$ x-int: $(-2, 0)$ y-int: $(0, 4)$

  Shade all y-values above the line.

- $2x - 3y \leq -6$

  Graph a solid line for the equation $2x - 3y = -6$

  Slope: $m = \frac{2}{3}$ x-int: $(-3, 0)$ y-int: $(0, 2)$

  Shade all y-values above the line.

Determine shading by (1) solving for y or (2) testing a point:

(1) $y \geq \frac{2}{3}x + 2$

(2) Test $(0, 0)$ which is below line:

$2(0) - 3(0) \leq -6$

$0 \leq -6$

False → shade above.

Your Turn. Graph the inequality.

48. $3x - 4y > 12$

49. $x - y \geq 2$ *and* $x \geq 3$

50. $3x + 2y > 6$ *or* $x - 2y > 2$

51. $x - y \geq 1$ *and* $x + y < 3$

52. $|x + 2| > 4$

Answers to Your Turn Problems

1. $20+5m+12n+3mn$

$$=(20+5m)+(12n+3mn)$$
$$=5(4+m)+3n(4+m)$$
$$=(4+m)(5+3n)$$

2. $y^2+7y-30$

$$=(y+10)(y-3)$$

3. $20x^2+47x+24$

$$=(4x+3)(5x+8)$$

4. $25x^2-90x+81$

$$=(5x)^2-2(5)(9)x+(9)^2=(5x-9)^2$$

5. $24x^2+42x+15$

$$=(6x+3)(4x+5)$$

6. p^4-10p^2+16

$$=\left(p^2\right)^2-10p^2+16=\left(p^2-2\right)\left(p^2-8\right)$$

7. x^4-81

$$=\left(x^2\right)^2-(9)^2=\left(x^2+9\right)\left(x^2-9\right)=\left(x^2+9\right)(x+3)(x-3)$$

8. x^5+3x^4-x-3

$$=\left(x^5+3x^4\right)+(-x-3)$$
$$=x^4(x+3)+(-1)(x+3)$$
$$=(x+3)\left(x^4-1\right)$$
$$=(x+3)\left(x^2+1\right)\left(x^2-1\right)$$
$$=(x+3)\left(x^2+1\right)(x+1)(x-1)$$

9. $2a^3+a^2-14a-7$

$$=\left(2a^3+a^2\right)+\left(-14a-7\right)$$
$$=a^2\left(2a+1\right)+\left(-7\right)\left(2a+1\right)$$
$$=\left(2a+1\right)\left(a^2-7\right)$$

10. $p^2+15p+56$

$$=\left(p+8\right)\left(p+7\right)$$

11. $27z^2+42z-5$

$$=\left(9z-1\right)\left(3z+5\right)$$

12. $14c^2-17cd-6d^2$

$$=\left(7c+2d\right)\left(2c-3d\right)$$

13. $6a^3+12a^2-90a$

$$=6a\left(a^2+2a-15\right)=6a\left(a+5\right)\left(a-3\right)$$

14. $100a^2-9b^2$

$$=\left(10a\right)^2-\left(3b\right)^2=\left(10a+3b\right)\left(10a-3b\right)$$

15. $16x^3+32x^2-9x-18$

$$=\left(16x^3+32x^2\right)+\left(-9x-18\right)$$
$$=16x^2\left(x+2\right)+\left(-9\right)\left(x+2\right)$$
$$=\left(x+2\right)\left(16x^2-9\right)$$
$$=\left(x+2\right)\left(4x+3\right)\left(4x-3\right)$$

16. x^9-1

$$=\left(x^3\right)^3-1^3$$
$$=\left(x^3-1\right)\left(\left(x^3\right)^2+\left(1\right)\left(x^3\right)+\left(1\right)^2\right)$$
$$=\left(x^3-1\right)\left(x^6+x^3+1\right)$$
$$=\left(x-1\right)\left(x^2+x+1\right)\left(x^6+x^3+1\right)$$

17. $9x^4y^8w^6$

18. $\left(\frac{2}{x^5y^8}\right)^3 = \frac{8}{x^{15}y^{24}}$

19. $\frac{4^{-2}x^{-6}y^{-2}\cdot 2^5w^5}{2^3x^3y^3\cdot 6^{-2}w^{-2}y^{-2}} = \frac{9w^7}{x^9y^3}$

20. $\frac{c(c+2)}{(c+2)(c-2)}\cdot\frac{(c-2)^2}{c(c-1)} = \frac{c-2}{c-1}$

21. $\frac{z(z+2)}{5+z}\cdot\frac{3(z-2)}{(2-z)(2+z)} = \frac{3z(z-2)}{(5+z)(2-z)} = -\frac{3z}{5+z}$

22. $\frac{15+2(x+3)+5x}{x(x+3)} = \frac{7x+21}{x(x+3)} = \frac{7(x+3)}{x(x+3)} = \frac{7}{x}$

23. $\frac{4(x^2-x+1)+1(x+1)-12}{(x+1)(x^2-x+1)} = \frac{4x^2-3x-7}{(x+1)(x^2-x+1)} = \frac{(4x-7)(x+1)}{(x+1)(x^2-x+1)} = \frac{4x-7}{x^2-x+1}$

24. $\frac{\left(\frac{x}{y}-\frac{16y}{x}\right)}{\left(\frac{1}{y}-\frac{4}{x}\right)}\cdot\frac{xy}{xy} = \frac{x^2-16y^2}{x-4y} = \frac{(x+4y)(x-4y)}{x-4y} - x+4y$

25 $\frac{\frac{x}{x-2}+1}{\frac{3}{x^2-4}+1} = \frac{\frac{x+x-2}{x-2}}{\frac{3+x^2-4}{x^2-4}} = \frac{2x-2}{x-2}\cdot\frac{x^2-4}{x^2-1} = \frac{2(x-1)}{x-2}\cdot\frac{(x+2)(x-2)}{(x+1)(x-1)} = \frac{2(x+2)}{x+1}$

26 $\frac{1-\frac{5}{x}-\frac{24}{x^2}}{1-\frac{15}{x}+\frac{56}{x^2}} = \frac{x^2-5x-24}{x^2-15x+56} = \frac{(x-8)(x+3)}{(x-8)(x-7)} = \frac{x+3}{x-7}$

27. $-12 = x^2+8x$ and $x \neq 0 \rightarrow x^2+8x+12=0 \rightarrow (x+6)(x+2)=0 \rightarrow \{-6, -2\}$

28. $3(2x+1)-x = 2x(2x+1)$ and $x \neq 0, -\frac{1}{2}$

$\rightarrow 6x+3-x = 4x^2+2x \rightarrow 4x^2-3x-3=0 \rightarrow x = \frac{3\pm\sqrt{9+48}}{8} = \left\{\frac{3\pm\sqrt{57}}{8}\right\}$

29. $4x-7=\pm 9 \rightarrow x=\frac{7\pm 9}{4} \rightarrow x=\left\{-\frac{1}{2},\ 4\right\}$

30. $(2x-3)(x-1)=0 \rightarrow x=\left\{1,\ \frac{3}{2}\right\}$

31. $(x^2-25)(x^2-4)=0 \rightarrow x^2=25 \text{ or } x^2=4 \rightarrow x=\{\pm 5,\ \pm 2\}$

32.
$$(2x+5)^2-4(2x+5)-6=0 \ \rightarrow u=2x+5$$
$$u^2-4u-6=0 \rightarrow u^2-4u+4=6+4 \rightarrow (u-2)^2=10 \rightarrow u=2\pm\sqrt{10}$$
$$2x+5=2\pm\sqrt{10} \ \rightarrow x=\left\{\frac{-3\pm\sqrt{10}}{2}\right\}$$

33.
$$(4m^{2/3}-9)(m^{2/3}-1)=0 \rightarrow m^{2/3}=\frac{9}{4} \text{ or } m^{2/3}=1$$
$$\rightarrow m^{1/3}=\pm\frac{3}{2} \text{ or } m^{1/3}=\pm 1 \rightarrow m=\pm\frac{27}{8} \text{ or } m=\pm 1 \rightarrow m=\left\{\pm\frac{27}{8},\ \pm 1\right\}$$

34.
$$9x^2=16-10x \rightarrow 9x^2+10x-16=0 \rightarrow (9x-8)(x+2)=0 \rightarrow x=\frac{8}{9} \text{ or } x=-2$$
$$3\left(\frac{8}{9}\right)\stackrel{?}{=}\sqrt{16-10\left(\frac{8}{9}\right)} \rightarrow \frac{8}{3}=\sqrt{\frac{64}{9}} \ \ \textit{yes}$$
$$3(-2)\stackrel{?}{=}\sqrt{16-10(-2)} \rightarrow -6=\sqrt{36} \ \ \textit{no} \rightarrow x=\left\{\frac{8}{9}\right\}$$

35. $3x-5=\pm\sqrt{12} \ \rightarrow x=\left\{\frac{5\pm 2\sqrt{3}}{3}\right\}$

36. $12+3(g^2-2g)=6g \text{ and } g\neq 0,\ 2 \rightarrow 3g^2-12g+12=0 \rightarrow \ 3(g-2)^2=0 \rightarrow g=2! \rightarrow \varnothing$

37. $x^2+8x-9=x^2+2x+1 \rightarrow 6x=10 \rightarrow x=\left\{\frac{5}{3}\right\}$

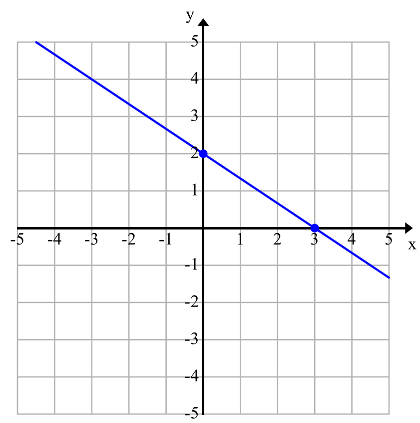
38.
$$x-6=9 \ \rightarrow x=15$$
$$\sqrt{15-6}+2=\sqrt{9}+2=3+2=5 \rightarrow x=\{15\}$$

39. $d=\sqrt{(-3-0)^2+(2-(-4))^2}=\sqrt{9+36}=\sqrt{45}=3\sqrt{5}$

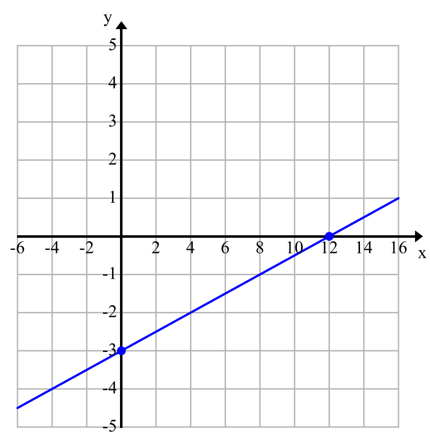
40. $13=\sqrt{(x-1)^2+(-3-2)^2} \rightarrow (x-1)^2+25=169 \rightarrow x=1\pm\sqrt{144} \rightarrow (13,\,-3),\,(-11,\,-3)$

41. $\left(\dfrac{x-4}{2},\,\dfrac{y+5}{2}\right)=(-1,\,3) \rightarrow \dfrac{x-4}{2}=-1 \text{ and } \dfrac{y+5}{2}=3 \rightarrow P_2=(2,\,1)$

42. $m=-\dfrac{2}{3},\; x-\text{int}:(3,\,0),\; y-\text{int}:(0,\,2)$



43. $m=\dfrac{1}{4},\; x-\text{int}:(12,\,0),\; y-\text{int}:(0,\,-3)$



44. $m=\dfrac{-6-3}{2-(-4)}=\dfrac{-9}{6}=-\dfrac{3}{2}$

$3=-\dfrac{3}{2}(-4)+b \rightarrow 3=6+b \rightarrow b=-3 \rightarrow y=-\dfrac{3}{2}x-3 \rightarrow \dfrac{3}{2}x+y=-3 \rightarrow 3x+2y=-6$

45. (a) $m_1=\dfrac{2}{3} \rightarrow m_{\parallel}=\dfrac{2}{3}$

$6=\dfrac{2}{3}(-4)+b \rightarrow b=6+\dfrac{8}{3}=\dfrac{26}{3} \rightarrow y=\dfrac{2}{3}x+\dfrac{26}{3}$

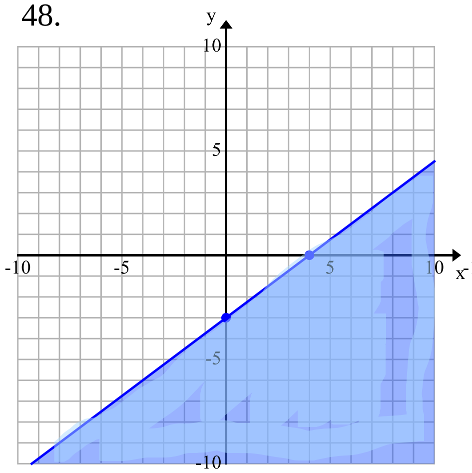
(b) $m_1=-\dfrac{3}{5} \rightarrow m_{\perp}=\dfrac{5}{3}$

$6=\dfrac{5}{3}(-4)+b \rightarrow b=6+\dfrac{20}{3}=\dfrac{38}{3} \rightarrow y=\dfrac{5}{3}x+\dfrac{38}{3}$

46. $\left(\frac{9}{7}, \frac{17}{7}\right)$

47. $(5, 3)$

48.



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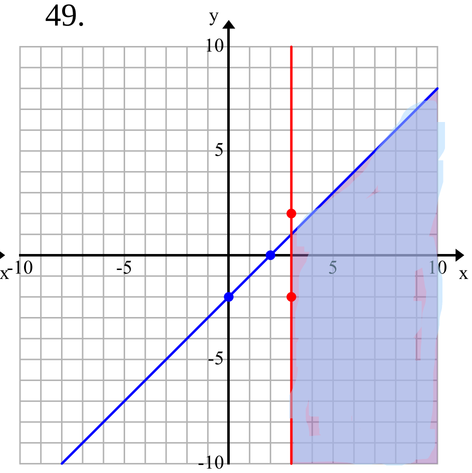

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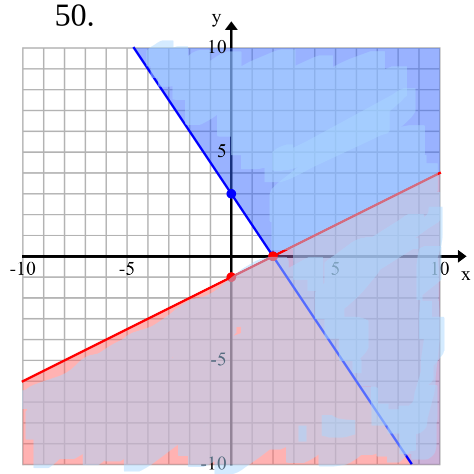

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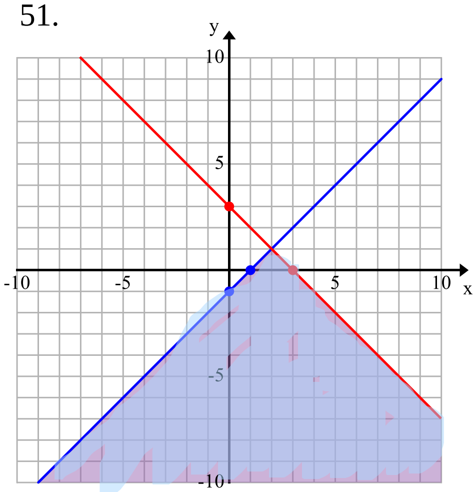

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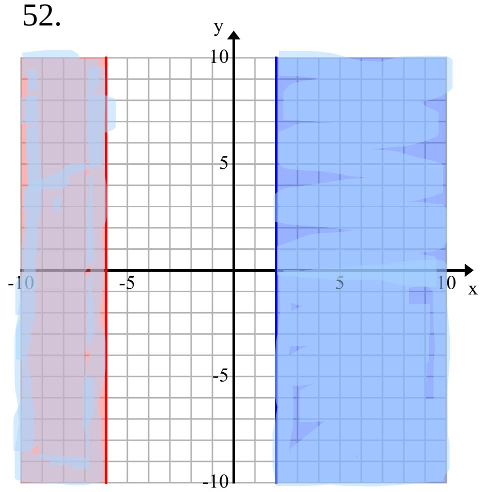

Additional Resources

https://www.khanacademy.org/math/algebra-home

YouTube - Patrick JMT